NATURAL AND TECHNICAL SCIENCES

UDC 514.1/512.64/530.12

MSC2020: 15A/51F/53B30/83-02

DOI 10.24412/2541-9285-2026-1106-157-199

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EQUIVALENT ACTIONS OF POTENTIALS FROM ACCELERATION AND GRAVITY ON TIME AND WORLD LINES IN MINKOWSKI SPACE-TIME INSTEAD OF ITS CURVING

Abstract

In the article, based on the new math subject "Tensor Trigonometry" from its author-himself, it is established that a long-standing problem of inconsistency between Theory of Relativity and Relativistic Quantum Mechanics in presence of gravity is artificial. We have revealed the true nature of curving actions, but in Minkowski space-time, by inner acceleration and/or intensity of gravity – both differentially equivalent. With potential Π_0 of the Universe, we discussed some actual topical issues about the origin of light speed "c" and – whether it is a constant in the Universe, and also about the genesis of mass energy as $E=mc^2$.

Key words: tensor trigonometry, symmetric and general tensors of motions, polar decomposition, tensor of axial rotations, symmetric and general tensor of energy and momentum, Pythagorean theorem for three momenta, hyperbolic geometry, angular deviations, Thomas precession, theory of relativity, Poincaré group, reflector-tensor, time-like orthoprocession, summing 4- and 3-velocities, Pythagorean theorem for 4- and 3-accelerations, time dilations of cosine and potential nature, GR-effects, the Universe potential, light speed, energy of mass.

Chapter 1. Introduction

In recent decades, another fundamental crisis has emerged again in theoretical physics after numerous but always unsuccessful attempts to reconcile Einstein's General Theory of Relativity (GTR) with Dirac's Relativistic Quantum Mechanics (RQM). The cause of the crisis, long recognized by the perceptive relativists, is the inconsistency of the curved 4D space-time of GTR with the conditions of Noether's Theorem [1] for the energy-momentum conservation Law and the Laws of RQM, but to which the 4D space-time Poincaré Q_c^{3+1} and Minkowski P^{3+1} correspond. The latter also used in Dirac's RQM [4] and in the Higgs' theory of matter inertia [5]. Einstein's dream since 1931 of a final unified theory of all four interactions (including gravity) then logically came into conflict with the Gödel's Incompleteness Theorem. The relentless attempts of numerous enthusiasts of this hypothetical theory, even with abundant grant support, became in GTR akin to solving the problems of "squaring circle" and "perpetuum mobile". As seem that the obvious mathematical inconsistency between the curved and flat continua of GTR and RQM is quite evident. However, in this stagnant problem, the conservatism of a very influential part of the relativistic community, adherents of the *geometric* GTR and which has formed over more than 100 years, has played an inertial negative role, becoming increasingly esoteric and aggressive. So, in order to defend it, stubborn apologists for GTR even introduce *ad hoc* new and clearly redundant concepts!? Although, upon its appearance, GTR itself immediately eliminated the relativistic Lorentz transformations and the Lobachevsky 3D hyperspace [6] for summing up the principal hyperbolic motions, their angles, curvatures, relativistic velocities, inner accelerations and much more!

However, P^{3+1} is using naturally in the Theory of Relativity (TR), RQM, and relativistic Maxwell's electromagnetism. This brilliant reconciling idea was first proposed and implemented by Rosen, Einstein's assistant and colleague at Princeton University [7]. However, the most developed theory using two metric tensors was the Logunov's Relativistic Theory of Gravity [8] with its classical field approach to gravity describing motions in an *observable* pseudo-Riemannian space-time. Bimetric theories apply the same absolute tensor calculus with covariant differentiation for describing motions and all GR-effects.

Sometimes the solution to a stagnant problem lies without complications and somewhere at its source. The author approached its solution in an unusual way – along with applying a new mathematical subject, first published by him in 2004 in the monograph "Tensor Trigonometry" [9], with its consistent presentation and primary use in geometric and physical fields, including TR and RQM. In its English and latest 3rd edition by the author "Tensor Trigonometry" [10] – the most expanded and updated, it was published in early 2025. The math subject is intended, for example, for descriptive analysis in homogeneous and isotropic binary spaces and on their *perfect* hypersurfaces of radius parameter "R" with non-Euclidean geometries. On such perfect surfaces, the summary motions are described by also angular increments or differentials. Tensor Trigonometry uses simple tensor and vector analysis with simplest *orthogonal differentiation and integration*! However, with Tensor Trigonometry approach, the Laws of TR and GR-effects are explained without distortion of space-time, either in the original complex-valuated binary quasi-Euclidean space-time of Poincaré Q_c^{3+1} with an imaginary time-arrow {ict}, pointing to the future, or in the realificated from it pseudo-Euclidean space-time of Minkowski P^{3+1} with a real time-arrow.

Chapter 2. Non-Euclidean hyperbolic, Euclidean orthospherical, and general pseudo-Euclidean tensors of discrete transformations or rotations.

In Poincaré TR [2], all four coordinates of a material object *N* barycenter in initial base E_1 and next E are expressed in trigonometric form as a passive or active pure rotation at an angle iγ in complex Q_c^{3+1} or pure hyperbolic motion iγ on its imaginary Lobachevsky hypersurface (top sheet of hyperboloid II in P^{3+1}), identical to the homogeneous Lorentz transformations [11], named so by Poincaré for his homogeneous and isotropic space-time with its Euclidean metric tensor $\{I^+\}$ and reflector tensor $\{I^{+-}\}$[9, p.154]; [10, p. 137 (cosα=±1):

$$\begin{bmatrix} \cos i\gamma & 0 & 0 & \pm\sin i\gamma\cdot\cos\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mp\sin i\gamma\cdot\cos\alpha & 0 & 0 & \cos i\gamma \end{bmatrix}\cdot\begin{vmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ ict^{(1)} \end{vmatrix} = \begin{vmatrix} \cosh\gamma\cdot x_1^{(1)} \mp \sinh\gamma\cdot\cos\alpha\cdot ct^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ i[\cosh\gamma\cdot ct^{(1)} \mp \sinh\gamma\cdot\cos\alpha\cdot x_1^{(1)}] \end{vmatrix} = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ ict \end{vmatrix}.$$

In Poincaré interpretation (1905), the fourth coordinate is imaginary and reduced to common Euclidean metric by the factor "ic" (speed of light in a vacuum). This gave his 4D space-time homogeneity and isotropy, and velocity v its maximum "c", which is Einstein's Postulate in his version of STR [12], but as a consequence from max{tanh γ}=1. The same action, but in 4D form, is expressed by our 4x4-tensor of motion roth Γ=F(γ, $\mathbf{e}_\alpha$) [9, p. 232] or [10, p. 202] in the Minkowski space-time P^{3+1} with its identical metric and reflector tensor $\{I^{+-}\}$:

$$roth\ \Gamma = \left[\begin{array}{c|c} I_{3\times3} + (\cosh\gamma - 1)\cdot \mathbf{e}_\alpha\mathbf{e}'_\alpha & \sinh\gamma\cdot\mathbf{e}_\alpha \\ \hline \sinh\gamma\cdot\mathbf{e}'_\alpha & \cosh\gamma \end{array}\right] = \left[\begin{array}{c|c} \overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} + \cosh\gamma\cdot\overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} & \sinh\gamma\cdot\mathbf{e}_\alpha \\ \hline \sinh\gamma\cdot\mathbf{e}'_\alpha & \cosh\gamma \end{array}\right], \qquad (1)$$

where $\overleftarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} = \mathbf{e}_\alpha\mathbf{e}'_\alpha$ and $\overrightarrow{\mathbf{e}_\alpha\mathbf{e}'_\alpha} = I_{3\times3} - \mathbf{e}_\alpha\mathbf{e}'_\alpha$ are *eigenprojectors* for $\mathbf{e}_\alpha$ in $\mathcal{E}^3$ [9].

With two-steps non-collinear hyperbolic rotations or motions in P^{3+1}, an additional orthospherical rotation Θ arises, identified in 1914 by Silberstein as a kinematic effect of TR [13]. It appears in the base E_1 in canonical form in our general pseudo-Euclidean 4x4-tensor of rotation or motion and in its polar decomposition into hyperbolic rothΓ and orthospherical rotΘ parts [10, p. 231]. These two rotations in all their combinations give us a complete and continuous Lorentz group (!). In (n+1)-D binary spaces and on their perfect hypersurfaces of radius-parameter R, their n-D rotations and motions are isomorphic (!). For two- and k-steps non-collinear rotations or motions, but and even for integral rotations or motions, with the same metric and reflector tensor $\{I^{+-}\}$, we have:

$$roth\{\Gamma,\Theta\} = roth\ \Gamma\cdot\overrightarrow{rot}\ \Theta_{4\times4} = \prod_{k=1}^{s} roth\ \Gamma_k = \left[\begin{array}{c|c} [\overrightarrow{rot}\ \Theta]_{3\times3} + (\cosh\gamma-1)\cdot\mathbf{e}_\sigma\mathbf{e}'_{\angle\sigma} & \sinh\gamma\cdot\mathbf{e}_\sigma \\ \hline \sinh\gamma\cdot\mathbf{e}'_{\angle\sigma} & \cosh\gamma \end{array}\right] = \qquad (2)$$

$$= \left[\begin{array}{c|c} I_{3\times3} + (\cosh\gamma-1)\cdot\mathbf{e}_\sigma\mathbf{e}'_\sigma & \sinh\gamma\cdot\mathbf{e}_\sigma \\ \hline \sinh\gamma\cdot\mathbf{e}'_\sigma & \cosh\gamma \end{array}\right]\cdot\left[\begin{array}{c|c} [\overrightarrow{rot}\ \Theta]_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \end{array}\right];\ \mathbf{e}_\sigma\mathbf{e}'_\sigma = \overleftarrow{\mathbf{e}_\sigma\mathbf{e}'_\sigma},\quad \mathbf{e}_\sigma\mathbf{e}'_{\angle\sigma} = \cos\theta\cdot\overleftarrow{\mathbf{e}_\sigma\mathbf{e}'_{\angle\sigma}}. \qquad (3)$$

Γ and γ is the angle of hyperbolic rotation around the frame axis {ct} in TR or the ordinate axis {y} in the pseudo-Euclidean space P^{3+1} at a final direction $\mathbf{e}_\sigma$. The hyperbolic angle γ determines orthoprojections of Poincaré 4-velocity "**c**" onto subspace $E^{3(1)}$ as the 3-velocities of object *N* in the base E_1 – namely, as *proper velocity* $v^*=dx/d\tau=c\cdot\sinh\gamma$ and *coordinate velocity* $v=dx/dt=c\cdot\tanh\gamma < c$.

Θ и θ is the angle of orthospherical axial rotation rotΘ around the 3rd normal axis $\mathbf{e}_\mu$ in *the sine normal Euclidean plane:*

$\mathcal{E}^2_{Ns} \equiv \langle \mathbf{e}_\alpha, \mathbf{e}_\nu \rangle \equiv \langle \mathbf{e}_\sigma, \mathbf{e}_{\angle\sigma} \rangle \subset \mathcal{E}^3.$

Orthospherical *orbital rotations* of object *N* with mass m at its velocity $\mathbf{v}_\alpha$ are realized by rotations of a world line, also by the tensor rotΘ, namely: by its tangent $\mathbf{i}_\alpha$ and its pseudonormal $\mathbf{p}_\alpha$ around the normal axes $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ in $E^{3(1)}$ or $E^{3(m)}$ under action of normal accelerations g_k. Orthospherical *axial rotations* of a body *N* with momentum of inertia J_0 and angular velocity $\mathbf{w}_\alpha$ are possible in its motion and rest around three normal axes *in* $E^{3(1)}$ or $E^{3(m)}$ only under actions of their torques P_k, except for the *Thomas precession* – see further. The orbital and axial momenta of rotation of object *N* around its axes have a common nature and physical dimension. Each similar rotation has 3 degrees of freedom with axes $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$. With the first $\mathbf{e}_\alpha$ it gives as if the "*Cardano gimbal*" in $E^{3(1)}$ to *the non-relativistic binormal plane* $E^{2(1)} \equiv\langle \mathbf{e}_\mu, \mathbf{e}_\nu \rangle$. In (2), (3) orthospherical rotations $\mathrm{rot}\Theta_{3x3}$ and $\mathrm{rot}\Theta_{4x4}$ act in $E^3 \subset P^{3+1}$ as Euclidean ones. All 7 rotations-motions above and 3 translations form the *complete Poincaré group* ‹1+3+3+3=10›.

Below we will derive in general tensor-vector-scalar (tvs) form and with orthospherical part rotΘ in (2) and (3) the General Laws of summation of these space-like two-steps hyperbolic rotations-motions (1) in (2). The tensor of the 1st motion with $\mathbf{e}_\alpha$, applied to the tangent $\mathbf{i}_\beta$ at a world line of the 2nd motion (as also radius-vector $\mathbf{i}_\beta$ of trigonometric hyperboloid II with radius-parameter R=1 in P^{3+1}) – both in their canonical forms in E_1 give us all the laws of summation of two-steps rotations or motions, including even the elements in Lobachevsky hyperbolic geometry [6], as well as of 4- and 3-velocities in TR, in tensor and projective trigonometric forms (the latter in vector – sine and tangent, in scalar – cosine and secant) *in direct and reverse sequences of rotations-motions* [10, p. 293]:

$$roth\ \Gamma_{12} \cdot \mathbf{i}_{23} = \begin{array}{|c|c|}\hline I_{3\times3} + (\cosh\gamma_{12} - 1)\cdot \mathbf{e}_\alpha \mathbf{e}'_\alpha & +\sinh\gamma_{12}\cdot\mathbf{e}_\alpha \\ \hline +\sinh\gamma_{12}\cdot\mathbf{e}'_\alpha & \cosh\gamma_{12} \\ \hline\end{array} \cdot \left\{\frac{\sinh\gamma_{23}\cdot\mathbf{e}_\beta}{\cosh\gamma_{23}}\right\} =$$

$$= \left\{\frac{[\sinh\ \gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{23}\cdot(\cosh\gamma_{12}-1)]\cdot\mathbf{e}_\alpha + \sinh\gamma_{23}\cdot\mathbf{e}_\beta}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23}}\right\} =$$

$$= \left\{\frac{[\sinh\ \gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{23}\cdot\cosh\gamma_{12}]\cdot\mathbf{e}_\alpha + \sin\varepsilon\cdot\sinh\gamma_{23}\cdot\mathbf{e}_\nu}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23}}\right\} =$$

$$= \left\{\frac{\sinh\gamma_{13}\cdot\mathbf{e}_\sigma}{\cosh\gamma_{13}}\right\} = \mathbf{i}_{13} = \{\overrightarrow{rot}\ \Theta\}_{13}\cdot\overset{\angle}{\mathbf{i}}_{13} \to \mathbf{e}_\sigma = \overrightarrow{rot}\ \Theta_{3\times3}\cdot\mathbf{e}_{\overset{\angle}{\sigma}} \quad (\mathbf{i}'_{13}\cdot\{I^\pm\}\cdot\mathbf{i}_{13} = i^2 = -1). \qquad (4)$$

$$roth\ \Gamma_{23} \cdot \mathbf{i}_{12} = \begin{array}{|c|c|}\hline I_{3\times3} + (\cosh\gamma_{23} - 1)\cdot \mathbf{e}_\beta \mathbf{e}'_\beta & +\sinh\gamma_{23}\cdot\mathbf{e}_\beta \\ \hline +\sinh\gamma_{23}\cdot\mathbf{e}'_\beta & \cosh\gamma_{23} \\ \hline\end{array} \cdot \left\{\frac{\sinh\gamma_{12}\cdot\mathbf{e}_\alpha}{\cosh\gamma_{12}}\right\} =$$

$$= \left\{\frac{[\sinh\ \gamma_{23}\cdot\cosh\gamma_{12} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot(\cosh\gamma_{23}-1)]\cdot\mathbf{e}_\beta + \sinh\gamma_{12}\cdot\mathbf{e}_\alpha}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23}}\right\} =$$

$$= \left\{\frac{[\sinh\ \gamma_{23}\cdot\cosh\gamma_{12} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\cosh\gamma_{23}]\cdot\mathbf{e}_\beta + \sin\varepsilon\cdot\sinh\gamma_{12}\cdot\mathbf{e}_{\overset{\angle}{\nu}}}{\cosh\gamma_{12}\cdot\cosh\gamma_{23} + \cos\varepsilon\cdot\sinh\gamma_{12}\cdot\sinh\gamma_{23}}\right\} =$$

$$= \left\{\frac{\sinh\gamma_{13}\cdot\mathbf{e}_{\overset{\angle}{\sigma}}}{\cosh\gamma_{13}} = \right\} = \overset{\angle}{\mathbf{i}}_{13} = \{\overrightarrow{rot}'\ \Theta\}_{13}\cdot\mathbf{i}_{13} \to \mathbf{e}_{\overset{\angle}{\sigma}} = \overrightarrow{rot}'\ \Theta_{3\times3}\cdot\mathbf{e}_\sigma \quad (\overset{\angle}{\mathbf{i}}{}'_{13}\cdot\{I^\pm\}\cdot\overset{\angle}{\mathbf{i}}_{13} = i^2 = -1). \qquad (5)$$

In TR, as above, the external orthospherical angle ε between segments 12 and 23 is used, but in non-Euclidean geometries, the internal angle A_{123}=π–ε is used for them. Above the forward and backward summation of rotations or motions are related by the contrary orthospherical rotations rotΘ. From (2)–(5) we find a new Euclidean unit vector $\mathbf{e}_\sigma$ for the summary hyperbolic motion in two variants (4) and (5). They are used to calculate the tensor $rot\Theta_{3x3}$ in (2) and (3) with the Silberstein axial shift at the angle Θ and θ, as well as below by the formulas of Orthospherical Trigonometry in E^3 for quasi- and pseudo-Euclidean geometry:

$$\overrightarrow{rot}\ \Theta_{4\times4} = \begin{array}{|c|c|}\hline \{\overrightarrow{rot}\ \Theta\}_{3\times3} & \mathbf{0} \\ \hline \mathbf{0}' & 1 \\ \hline\end{array} = \begin{array}{|c|c|c|c|}\hline \cos\theta + \frac{r_1^2}{1+\cos\theta} & -r_3 + \frac{r_1 r_2}{1+\cos\theta} & +r_2 + \frac{r_1 r_3}{1+\cos\theta} & 0 \\ \hline +r_3 + \frac{r_1 r_2}{1+\cos\theta} & \cos\theta + \frac{r_2^2}{1+\cos\theta} & -r_1 + \frac{r_2 r_3}{1+\cos\theta} & 0 \\ \hline -r_2 + \frac{r_1 r_3}{1+\cos\theta} & +r_1 + \frac{r_2 r_3}{1+\cos\theta} & \cos\theta + \frac{r_3^2}{1+\cos\theta} & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline\end{array}. \qquad (6)$$

In the absence of spherical rotΦ in Q^{3+1} and hyperbolic rothΓ in P^{3+1}, principal motion tensor (6) does not give precession and deviation of angles!

For two-steps motions Γ_{12}, Γ_{23}, and Γ_{23}, Γ_{12}, or for summing RothΓ_k with their q directional cosines, where q=1, 2, 3, we have $\mathbf{e}_\sigma=\{\cos\sigma_q\}$ for forward and reverse orders. As a result, in E^3 they give us the 3rd directed normal axis $\mathbf{e}_\mu$ of axial rotation with an addition (–sin θ):

$$\vec{\mathrm{r}}_N(\theta)=\overrightarrow{\begin{vmatrix} r_1 \\ r_2 \\ r_3 \end{vmatrix}}=\mathrm{e}_{\overset{\angle}{\sigma}}\times\mathrm{e}_\sigma=\det\begin{bmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \cos\sigma_1 & \cos\sigma_2 & \cos\sigma_3 \\ \cos\overset{\angle}{\sigma}_1 & \cos\overset{\angle}{\sigma}_2 & \cos\overset{\angle}{\sigma}_3 \end{bmatrix}=\overrightarrow{\begin{vmatrix} \cos\overset{\angle}{\sigma}_2\cos\sigma_3-\cos\overset{\angle}{\sigma}_3\cos\sigma_2 \\ \cos\overset{\angle}{\sigma}_3\cos\sigma_1-\cos\overset{\angle}{\sigma}_1\cos\sigma_3 \\ \cos\overset{\angle}{\sigma}_1\cos\sigma_2-\cos\overset{\angle}{\sigma}_2\cos\sigma_1 \end{vmatrix}}=-\sin\theta\cdot\vec{\mathrm{e}}_N.(7)$$

$$|\sin\theta|=||\vec{\mathbf{r}}_N(\theta)||=\sqrt{r_1^2+r_2^2+r_3^2}\,,\ tr\,[\overrightarrow{rot}\,\Theta]_{3\times3}=2(\cos\theta+1);\ \cos\theta=\mathbf{e}'_\sigma\mathbf{e}_{\overset{\angle}{\sigma}}=tr\,[\overrightarrow{rot}\,\Theta]_{3\times3}/2-1.$$

So, $det\{\mathrm{e}_{\overset{\angle}{\sigma}},\mathrm{e}_\sigma,\vec{\mathbf{r}}_N\}>0\rightarrow\theta<0$; $(\mathrm{e}_{\overset{\angle}{\sigma}},\mathrm{e}_\sigma,\vec{\mathbf{r}}_N)$ and $(\mathrm{e}_{\overset{\angle}{\sigma}},\mathrm{e}_\sigma,\vec{\mathrm{e}}_N)$ are the *left and right triplets.*

The orthospherical angle θ and increment dθ are given as counterclockwise, and here with the sign "–" (but the orthospherical angle ε between the unit vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ of 1st and 2nd hyperbolic motions is given by default with the sign "+"), for example, in $E^{3(k)}\subset P^{3+1}$ in TO, as well as in the hyperbolic Lobachevsky geometry. Rule 1: $\mathrm{sgn}\theta_{13}=-\mathrm{sgn}\varepsilon$ (!); but for two-steps spherical rotations in quasi-Euclidean and spherical geometries Rule 2: $\mathrm{sgn}\theta_{13}=+\mathrm{sgn}\varepsilon$. Additionally, with (4) and (5), we obtain the following relations in $E^3\subset P^{3+1}$:

$$\left.\begin{array}{l} \mathrm{e}_{\overset{\angle}{\sigma}}=\overrightarrow{rot}'\,\Theta_{3\times3}\cdot\mathrm{e}_\sigma=\{\mathrm{e}_{\overset{\angle}{\sigma}}\cdot\mathrm{e}'_\sigma\}\cdot\mathrm{e}_\sigma=\sec\theta\cdot\overleftarrow{\{\mathrm{e}_{\overset{\angle}{\sigma}}\cdot\mathrm{e}'_\sigma\}}\cdot\mathrm{e}_\sigma, \\ \mathrm{e}_\beta=\overrightarrow{rot}\,\mathcal{E}_{3\times3}\cdot\mathrm{e}_\alpha=\{\mathrm{e}_\beta\cdot\mathrm{e}'_\alpha\}\cdot\mathrm{e}_\alpha=\sec\varepsilon\cdot\overleftarrow{\{\mathrm{e}_\beta\cdot\mathrm{e}'_\alpha\}}\cdot\mathrm{e}_\alpha; \\ \overrightarrow{rot}'\,\Theta_{3\times3}\cdot\overleftarrow{\{\mathrm{e}_\sigma\cdot\mathrm{e}'_\sigma\}}\cdot\overrightarrow{rot}\,\Theta_{3\times3}=\overleftarrow{\{\mathrm{e}_{\overset{\angle}{\sigma}}\cdot\mathrm{e}'_{\overset{\angle}{\sigma}}\}}; \end{array}\right\}\left\{\begin{array}{l} \vec{\mathrm{r}}_N(\theta)=\mathrm{e}_{\overset{\angle}{\sigma}}\times\mathrm{e}_\sigma=-\sin\theta\cdot\vec{\mathrm{e}}_N, \\ \vec{\mathrm{r}}_N(\varepsilon)=\mathrm{e}_\beta\times\mathrm{e}_\alpha=+\sin\varepsilon\cdot\vec{\mathrm{e}}_N, \\ \cos\theta=\mathrm{e}'_\sigma\mathrm{e}_{\overset{\angle}{\sigma}},\quad\cos\varepsilon=\mathrm{e}'_\alpha\mathrm{e}_\beta. \end{array}\right\}\qquad(8)$$

Under using (4) and (5), in the apexes of a triangle on the trigonometric hyperboloid II (R=i), we get its Lambert angular defect $-\delta\theta=\delta S/R^2$ [10, p. 226]. Using their quasi-Euclidean analogs (by abstract spherical-hyperbolic analogy only for principal angles as $\gamma=i\cdot\varphi$), in the apexes of a triangle on our oriented trigonometric hyperspheroid (R=1), we get its Harriot angular excess $\delta\theta=\delta S/R^2$.

Now we explain: how in $Q^{(n+1)}$ and $P^{(n+1)}$ the complete sets of their admissible homogeneous principal tensor trigonometric motive transformations are defined – see in [9, p. 116], [10, p. 106].

Thus, in $P^{(n+1)}$ we define all motive transformations in (1)–(3) as follows:

$\mathrm{roth}\Gamma\cdot\{I^{+-}\}\cdot\mathrm{roth}\Gamma=\{I^{+-}\}=\mathrm{roth}(-\Gamma)\cdot\{I^{+-}\}\cdot\mathrm{roth}(-\Gamma)$.

Correspondingly, in $Q^{(n+1)}$ we define them so:

$\mathrm{rot}\Phi\cdot\{I^{+-}\}\cdot\mathrm{rot}\Phi=\{I^{+-}\}=\mathrm{rot}(-\Phi)\cdot\{I^{+-}\}\cdot\mathrm{rot}(-\Phi)$.

And admissible orthospherical transformations are defined there identically:

$\mathrm{rot}'\Theta\cdot\{I^{+-}\}\cdot\mathrm{rot}\Theta=\{I^{+-}\}=\mathrm{rot}(-\Theta)\cdot\{I^{+-}\}\cdot\mathrm{rot}'(-\Theta)$.

Note, that all *reflective tensor trigonometric transformations*, with reflective tensor angles, are defined also simply enough in [9], [10]. All these *motive and reflective tensors* in $P^{(3+1)}$ form the complete set of Lorentzian transformations!

Chapter 3. Kinematics and dynamics in Minkowski space-time.

We express in P^{3+1} 4x4-tensors of velocity $\mathbf{T}_V$, of momentum $\mathbf{T}_P$, of energy-momentum $\mathbf{T}_E$. All of them are proportional and, most importantly, to *our dimensionless* trigonometric motion tensor (1) in their canonical structures in the original base E_1, which finally clarifies their essence to relativists [10, p. 303]:

$$\mathcal{T}_V = c\cdot roth\Gamma = c\cdot\left[\begin{array}{c|c} I_{3\times3}+(\cosh\gamma-1)\cdot\overleftarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'} & \sinh\gamma\cdot\mathbf{e}_\alpha \\ \hline \sinh\gamma\cdot\mathbf{e}'_\alpha & \cosh\gamma\end{array}\right] = \left[\begin{array}{c|c} c\cdot I_{3\times3}+(c^*-c)\cdot\overleftarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'} & \mathbf{v}^*_\alpha \\ \hline \mathbf{v}^{*\prime}_\alpha & c^*\end{array}\right], \quad (9)$$

$$\mathcal{T}_P = m_0c\cdot roth\Gamma = P_0\cdot\left[\begin{array}{c|c} I_{3\times3}+(\cosh\gamma-1)\cdot\overleftarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'} & \sinh\gamma\cdot\mathbf{e}_\alpha \\ \hline \sinh\gamma\cdot\mathbf{e}'_\alpha & \cosh\gamma\end{array}\right] = \left[\begin{array}{c|c} P_0\cdot I_{3\times3}+\Delta P\cdot\overleftarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'} & \mathbf{p}_\alpha \\ \hline \mathbf{p}'_\alpha & P\end{array}\right], \quad (10)$$

$$\mathcal{T}_E = m_0c^2\cdot roth\Gamma = E_0\cdot\left[\begin{array}{c|c} I_{3\times3}+(\cosh\gamma-1)\cdot\overleftarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'} & \sinh\gamma\cdot\mathbf{e}_\alpha \\ \hline \sinh\gamma\cdot\mathbf{e}'_\alpha & \cosh\gamma\end{array}\right] = \left[\begin{array}{c|c} E_0\cdot I_{3\times3}+A\cdot\overleftarrow{\mathbf{e}_\alpha\cdot\mathbf{e}_\alpha{}'} & \mathbf{p}_\alpha c \\ \hline \mathbf{p}'_\alpha c & E\end{array}\right], \quad (11)$$

$$\Rightarrow\ E = mc^2 = \cosh\gamma\cdot E_0 = E_0+(\cosh\gamma-1)\cdot E_0 = E_0+k_E\cdot E_0 = E_0+A,\ \text{где}\ k_E = \cosh\gamma-1 = \Delta E/E_0.\ (11-A)$$

$$\mathbf{P_0} = P_0\cdot\mathbf{i}_\alpha = m_0\cdot\mathbf{c} = P_0\cdot\begin{bmatrix}\mathbf{sinh}\ \gamma\\ \cosh\gamma\end{bmatrix} = m_0c\cdot\begin{bmatrix}\sinh\ \gamma\cdot\mathbf{e}_\alpha\\ \cosh\gamma\end{bmatrix} = \begin{bmatrix}m_0\mathbf{v}^*_\alpha\\ m_0c^*\end{bmatrix} = \begin{bmatrix}m\mathbf{v}_\alpha\\ mc\end{bmatrix} = \begin{bmatrix}\mathbf{p}_\alpha\\ P\end{bmatrix} = \begin{bmatrix}\mathbf{p}_\alpha\\ E/c\end{bmatrix}.\ (12)$$

$\mathbf{T}_V$ includes the Poincaré 4x1-velocity i**c** of an object along its world line in P^{3+1}. According to its pseudo-Euclidean modulus $|\mathrm{i}\mathbf{c}|_P=c$, it is constant for any matter.

$\mathbf{T}_P$ includes its own 4x1-momentum $\mathbf{P}_0$ along its world line in P^{3+1}.

$\mathbf{T}_E$ includes the scalar total energy $E=mc^2$ and the work $A=\Delta E$ precisely in the direction of motion $\mathbf{e}_\alpha$ initially or $\mathbf{e}_\sigma$ along the way of motion.

$\mathbf{P}_0=m_0\mathbf{c}$ is the hypotenuse of an internal pseudo-Euclidean right triangle (inside of the light cone) of three momenta with two cathetus-momenta as *total* along time-arrow {ct} $P=mc=m_0c^*$ and *real* $p=mv=m_0v^*$ in Euclidean subspace $E^{3(1)}$.

In P^{3+1} *the Absolute Pseudo-Euclidean Pythagorean Theorem of 3 momenta* acts

$$\mathbf{P}_0 = m_0 \cdot \mathbf{c} = P_0 \cdot \mathrm{i} = P \cdot \mathrm{i}_1 + p \cdot \mathrm{j} \ \Rightarrow (iP_0)^2 = (iP)^2 + p^2 = -P_0^2 = -P^2 + p^2 < 0 \text{ (at tensor } I^{\pm}). \quad (13)$$

See also its proportional trigonometric progenitor $1=\cosh^2\gamma-\sinh^2\gamma$ also for velocities c2=c*2–v*2, with two Euclidean orthoprojections of 4x1-velocity as c* – supervelocity *of the orthoprocession of a world line* with an object *N* parallel to time-arrow {ct} and v* – the proper velocity of *N* in proper time τ.

The angle Γ, γ in trigonometric motion tensors (1), (2), (5) and in physical tensors (9)–(11) is common too. For instance, taking into account (5), (12):

$$roth\,\Gamma_{23} \cdot \mathbf{P}_{0\alpha} = \begin{array}{|c|c|}\hline I_{3\times3} + (\cosh\gamma_{23} - 1) \cdot \mathrm{e}_\beta \mathrm{e}'_\beta & +\sinh\gamma_{23} \cdot \mathrm{e}_\beta \\ \hline +\sinh\gamma_{23} \cdot \mathrm{e}'_\beta & \cosh\gamma_{23} \\ \hline \end{array} \cdot \begin{bmatrix} \mathrm{p}_\alpha \\ E/c \end{bmatrix} = P_0 \cdot \left\{ \frac{\sinh\gamma_{13} \cdot \mathrm{e}_{\angle\sigma}}{\cosh\gamma_{13}} \right\} = \begin{bmatrix} \mathrm{p}_{\angle\sigma} \\ E'/c \end{bmatrix} = \mathbf{P}_{0\angle\sigma}. \quad (14)$$

Consequently, in P^{3+1} the Lorentz transformations act identically on both the 4 coordinates of an object and its momenta, energy, velocities, but in (14) actively, i.e., oppositely to (4) – with a shift of +θ. They also act identically on Maxwell's electromagnetism, as proved by Lorentz in 1904 [11] and leading Poincaré to relativity theory in June 1905! From our tensors, it can be clarity seen that tensor trigonometric and physical formulas in the space-time of Poincaré Q_c^{3+1} and in the Minkowski space-time P^{3+1} are proportional to each other with their own constant physical and geometric factors as isomorphism (only under v<c)!

For the Einstein's photons in $\mathbf{p}_\alpha = m_0\mathbf{c}_\alpha^* = m\mathbf{c}_\alpha$, these formulas first give an uncertainty of 0·∞, and then reveal it in the base E_1 through E=hν. With this in mind, we transform formulas (10) and (11) for the photon motion as follows:

$$T_P = m_0 c \cdot roth\Gamma = 0 \cdot \infty = \begin{array}{|c|c|}\hline P \cdot \overleftarrow{\mathrm{e}_\alpha \cdot \mathrm{e}_\alpha'} & \mathrm{p}_\alpha \\ \hline \mathrm{p}'_\alpha & P \\ \hline \end{array} \sim T_E = m_0 c^2 \cdot roth\Gamma = 0 \cdot \infty = \begin{array}{|c|c|}\hline E \cdot \overleftarrow{\mathrm{e}_\alpha \cdot \mathrm{e}_\alpha'} & \mathrm{p}_\alpha c \\ \hline \mathrm{p}'_\alpha c & E \\ \hline \end{array}. \quad (15)$$

Paradox is that these particles – massless, seemingly at rest, and massive in motion on an isotropic cone, due to the action of the normal field strength g_f, can, according to Newton's 2nd Law, change the direction of a light ray $\mathbf{e}_\alpha$, without changing its scalar velocity "c." This does not contradict the Theory of Reality, but causes Newtonian curvature of the light ray – see more further!

Let us reveal the *complete* absolute relativistic dynamics of object N on its world line in P^{3+1}, for example, as 4x4-tensor with the constant factor $E_0=m_0c^2$, which is now proportional to the *general* 4x4-tensor of rotations-motions (2), (3) as also 4x4-matrix of general homogeneous Lorentz transformations, expressed in their canonical form in the initial unity base E_1:

$$\mathcal{T}_E = E_0 \cdot roth\{\Gamma, \Theta\} = E_0 \cdot \left[\begin{array}{c|c} [\overrightarrow{rot}\ \Theta]_{3\times 3} + (\cosh\gamma - 1)\cdot \mathrm{e}_\sigma \mathrm{e}'_{\underset{\sigma}{\angle}} & \sinh\gamma \cdot \mathrm{e}_\sigma \\ \hline \sinh\gamma \cdot \mathrm{e}'_{\underset{\sigma}{\angle}} & \cosh\gamma \end{array}\right] = \qquad (16)$$

$$= c \cdot \left[\begin{array}{c|c} P_0 \cdot [\overrightarrow{rot}\ \Theta]_{3\times 3} + \Delta P \cdot \mathrm{e}_\sigma \mathrm{e}'_{\underset{\sigma}{\angle}} & \mathrm{p}_\sigma \\ \hline \mathrm{p}'_{\underset{\sigma}{\angle}} & P \end{array}\right] = \left[\begin{array}{c|c} E_0 \cdot [\overrightarrow{rot}\ \Theta]_{3\times 3} + A \cdot \mathrm{e}_\sigma \mathrm{e}'_{\underset{\sigma}{\angle}} & \mathrm{p}_\sigma c \\ \hline \mathrm{p}'_{\underset{\sigma}{\angle}} c & E \end{array}\right] = \qquad (17)$$

$$= E_0 \cdot roth\,\Gamma \cdot \overrightarrow{rot}\ \Theta_{4\times 4} = E_0 \cdot \left[\begin{array}{c|c} I_{3\times 3} + (\cosh\gamma - 1)\cdot \mathrm{e}_\sigma \mathrm{e}'_\sigma & \sinh\gamma \cdot \mathrm{e}_\sigma \\ \hline \sinh\gamma \cdot \mathrm{e}'_\sigma & \cosh\gamma \end{array}\right] \cdot \left[\begin{array}{c|c} [\overrightarrow{rot}\ \Theta]_{3\times 3} & 0 \\ \hline 0' & 1 \end{array}\right] = \qquad (18)$$

$$= c \cdot \left[\begin{array}{c|c} P_0 \cdot I_{3\times 3} + \Delta P \cdot \overleftarrow{\mathrm{e}_\sigma \mathrm{e}'_\sigma} & \mathrm{p}_\alpha \\ \hline \mathrm{p}'_\alpha & P \end{array}\right] \cdot \left[\begin{array}{c|c} [\overrightarrow{rot}\ \Theta]_{3\times 3} & 0 \\ \hline 0' & 1 \end{array}\right] = \left[\begin{array}{c|c} E_0 \cdot I_{3\times 3} + A \cdot \overleftarrow{\mathrm{e}_\sigma \mathrm{e}'_\sigma} & \mathrm{p}_\alpha c \\ \hline \mathrm{p}'_\alpha c & E \end{array}\right] \cdot \left[\begin{array}{c|c} [\overrightarrow{rot}\ \Theta]_{3\times 3} & 0 \\ \hline 0' & 1 \end{array}\right]. \qquad (19)$$

$$(E = mc^2,\ P = mc,\ \mathbf{p} = m\mathbf{v} = m_0\mathbf{v}^*;\ \overleftarrow{\mathrm{e}_\alpha \mathrm{e}'_\alpha} = \mathrm{e}_\alpha \mathrm{e}'_\alpha;\quad \overrightarrow{\mathrm{e}_\alpha \mathrm{e}'_\alpha} = I_{3\times 3} - \overleftarrow{\mathrm{e}_\alpha \mathrm{e}'_\alpha};\quad \mathrm{e}_\sigma \mathrm{e}'_{\underset{\sigma}{\angle}} = \cos\theta \cdot \overleftarrow{\mathrm{e}_\sigma \mathrm{e}'_{\underset{\sigma}{\angle}}},\quad \cos\theta = \mathrm{e}'_\sigma \mathrm{e}_{\underset{\sigma}{\angle}}.)$$

In the absence of external influences, (17) gives the conserving 4x4-tensor of energy and all momenta (principal and normal – orbital and axial) in P^{3+1}. Scalar momentum P=E/c in (13) influences on scalar time-like energy E of motion and on time dilation through cosine of motion angle. Normal momenta of object N are caused either from actions normal forces, or from actions of torques.

Moreover, axial momenta, with their momenta and energy in polar representation (19), are independent on the principal momentum $\mathbf{p}_\alpha=m_0\mathbf{v}_\alpha{}^*=m\mathbf{v}_\alpha$ with the hyperbolic rotations Γ, γ and on the orbital momenta with their orthospherical rotations. All momenta of axial rotations are naturally translated into the 3x3 Euclidean part of the initially unmixed 4x4-tensor of momentum. Note that *the angular elements* of these motion tensors with γ, coshγ, P and E *are independent of the decomposition operation* – see else further in Chapter 6!

We have once again become convinced that both the kinematics and the dynamics of all material objects N or particles are initially controlled by our dimensionless trigonometric tensors of motion in (1)-(3).

Chapter 4. Differential tangential, normal, and summary general rotations at a world line with proportional curvatures and inner accelerations.

For clarity, we represent Minkowski absolute space-time with the tensor $\{I^{+-}\}$ in a direct and hyperbolically orthogonal sum of two relative parts with angle Γ_k between them in the base E_k:

$$\langle \mathcal{P}^{3+1}\rangle \equiv \langle \mathcal{E}^3\rangle^{(k)} \boxtimes \overrightarrow{ct}^{(k)} \equiv \text{CONST}, \quad (n=3,\ q=1);\ \Delta ct^{(k)} > 0,\ (\text{in } \tilde{E}_1:\ k=1). \qquad (20)$$

In P^{3+1} the first hyperbolic differentials of motion and inner accelerations (proportional to the curvatures of a world line) obey proportional Pythagorean Theorems [10, p. 257]. If several inner accelerations act simultaneously on object *N*, then they are all summed geometrically in TR as Euclidean vectors. Thus, with the sine law of summation of two hyperbolic motions from (4) at $\gamma_{12}\to 0$ and $\gamma_{23}\to 0$, we obtain summary general angular hyperbolic differentials, for example, in their collinear ($\cos\varepsilon = \pm 1$) and in normal ($\cos\varepsilon = 0$) forms:

$$\sinh^2\gamma_{13} = \sinh^2\gamma_{12} + \sinh^2\gamma_{23} + (1+\cos^2\varepsilon)\cdot\sinh^2\gamma_{12}\cdot\sinh^2\gamma_{23} + 2\cos\varepsilon\cdot\cosh\gamma_{12}\cdot\sinh\gamma_{12}\cdot\cosh\gamma_{23}\cdot\sinh\gamma_{23}.$$

$$\to \sinh\gamma_{13} = \sinh\gamma_{12}\cdot\cosh\gamma_{23} \pm \cosh\gamma_{12}\cdot\sinh\gamma_{23} \ \Rightarrow\ d\gamma_{13} = d\gamma_{12} \pm d\gamma_{23} \to g_{13} = g_{12} \pm g_{23}\ \ (\cos\varepsilon = \pm 1).$$

$$\to \sinh^2\gamma_{13} = \sinh^2\gamma_{12} + (\cosh\gamma_{12}\cdot\sinh\gamma_{23})^2 \ \Rightarrow\ d\gamma_{13}^2 = d\gamma_{12}^2 + d\gamma_{23}^2 \to g_{13}^2 = g_{12}^2 + g_{23}^2\ \ (\cos\varepsilon = 0).$$

Most generally, for three independent accelerations at $\gamma_{12}\to 0$, $\gamma_{23}\to 0$, we have the same [10, p. 218]. The same Rule applies when decomposing an arbitrary relativistic or geometric motion into parallel and normal parts.

Minkowski hyperboloid II (upper sheet) with radius R=ic and with its Lobachevsky geometry [6], [14], represents the same vector-radius of the Poincaré 4-velocity $\mathbf{c}_\alpha = c\cdot\mathbf{i}_\alpha$ for object *N* on its world line in the tangential direction $\mathbf{i}_\alpha$. As a result of orthogonal differentiation of the 4-velocity with respect to imaginary time $di\tau$ along a world line, we obtain a usual pair of space-like inner accelerations: $\mathbf{g}_\alpha = c\cdot(d\gamma/d\tau)\cdot\mathbf{e}_\alpha$ as the principal parallel 4-acceleration in the direction of the 4-pseudonormal $\mathbf{p}_\alpha$ and $\mathbf{g}_\nu = c\cdot[\sinh\gamma\cdot(d\alpha_1/d\tau)]\cdot\mathbf{e}_\nu$ as the sine normal 3-acceleration in the direction of the 3-binormal $\mathbf{b}_\nu$ – both tangent to the hyperboloid II.

They sum up to the 4-acceleration also tangent to the hyperboloid II, with their Relative and Absolute Pythagorean Theorems in tensor-vector-scalar (tvs) form here in P^{2+1}. All they are proportional to the final differential $d\gamma_p$ and curvatures on the hyperboloid II as on *a perfect hypersurface* [10, p. 273]:

$$\left.\begin{array}{l} \cosh\gamma_p\, d\gamma_p \cdot \mathrm{e}_\beta = \cosh\gamma_p\,(\cos\varepsilon\, d\gamma_p\cdot \mathrm{e}_\alpha + \sin\varepsilon\, d\gamma_p \cdot \mathrm{e}_\nu) = \cosh\gamma_i\, d\gamma_i\cdot \mathrm{e}_\alpha + \sinh\gamma_i\, d\alpha_1 \cdot \mathrm{e}_\nu, \\ \cosh^2\gamma_p\, d\gamma_p^2 = \cos^2\varepsilon\,(\cosh^2\gamma_p\, d\gamma_p^2) + \sin^2\varepsilon\,(\cosh^2\gamma_p\, d\gamma_p^2) = \cosh^2\gamma_i\, d\gamma_i^2 + \sinh^2\gamma_i\, d\alpha_1^2; \\ \sinh\gamma_p\, d\gamma_p = \sinh\gamma_i\, d\gamma_i \quad \rightarrow d\gamma_p/d\gamma_i > 1. \end{array}\right\} \Rightarrow \qquad (21-I)$$

$$\Rightarrow \left\{\begin{array}{l} \cosh\gamma_p \cdot \mathbf{g}_\beta = \mathbf{g}^\star_\beta = \overline{\overline{\mathbf{g}^\star_\beta}} + \overset{\perp}{\mathbf{g}^\star_\beta} = \cosh\gamma_i \cdot g_\alpha \cdot \mathrm{e}_\alpha + v^* w^*_{\alpha_1} \cdot \mathrm{e}_\nu = g^\star_\alpha \cdot \mathrm{e}_\alpha + g_\nu \cdot \mathrm{e}_\nu, \\ \cosh^2\gamma_p \cdot g_\beta^2 = g_\beta^{\star 2} = (\overline{\overline{g^\star_\beta}})^2 + (\overset{\perp}{g^\star_\beta})^2 = \cosh^2\gamma_i \cdot g_\alpha^2 + (v^* w^*_{\alpha_1})^2 = g_\alpha^{\star 2} + g_\nu^2 = (c^*\eta^*)^2 + (v^* w^*_{\alpha_1})^2; \\ \sinh\gamma_p\, g_\beta = \sinh\gamma_i\, g_\alpha \quad \rightarrow g_\beta/g_\alpha > 1. \end{array}\right. \qquad (21-II)$$

$$\left.\begin{array}{l} d\gamma_p \cdot \mathrm{p}_\beta = d\gamma_i \cdot \mathrm{p}_\alpha + \sinh\gamma_i\, d\alpha_1 \cdot \mathrm{b}_\nu, \quad (\mathrm{p}'_\alpha \cdot I^\pm \cdot \mathrm{p}_\alpha = +1, \ \ \mathrm{b}'_\nu \cdot I^\pm \cdot \mathrm{b}_\nu = +1) \\ d\gamma_p^2 = d\gamma_i^2 + \sinh^2\gamma_i\, d\alpha_1^2 = \cos^2\varrho\, d\gamma_p{}^2 + \sin^2\varrho\, d\gamma_p{}^2 = \left(\overline{\overline{d\gamma_p}}\right)^2_P + \left(\overset{\perp}{d\gamma_p}\right)^2_E > 0. \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbf{g}_\beta = g_\alpha \mathrm{p}_\alpha + g_\nu \mathrm{b}_\nu, \\ g_\beta^2 = g_\alpha^2 + g_\nu^2; \end{array}\right. \qquad (22)$$

where we have: $d\gamma_p = d\lambda_R/R$, $\varrho > \varepsilon$, and $\mathbf{g}_\beta = g_\beta \mathbf{p}_\beta$ as a general internal 4-*acceleration.*

The tangent $\mathbf{i}_\alpha$ and the pseudonormal $\mathbf{p}_\alpha$ are related by symmetry with respect to the isotropic light cone. Therefore, they affect relativistic motion together. By partial differentiation of $\mathbf{p}_\alpha$ with respect to $d\tau$ (γ=const), we add else normal term $\mathbf{g}_\mu = c\cdot[\cosh\gamma\cdot(d\alpha_2/d\tau)]\cdot\mathbf{e}_\mu$ as the inner space-like cosine normal 3-acceleration along the 3-binormal $\mathbf{b}_\mu$ – now tangent to the hyperboloid I. The general Relative and Absolute Euclidean Pythagorean Theorems follows here in P^{3+1} with hypotenuses and 3 cathetus, for example, with accelerations [10, p. 281]:

$$\mathrm{g}_\Sigma = g_\Sigma \cdot \mathrm{p}_\Sigma = g_\alpha \mathrm{p}_\alpha + g_\nu \mathrm{b}_\nu + g_\mu \mathrm{b}_\mu \rightarrow g_\Sigma^2 = g_\alpha^2 + g_\nu^2 + g_\mu^2 = (c\eta^*)^2 + (v^* w^*_{\alpha_1})^2 + (c^* w^*_{\alpha_2})^2. \qquad (23)$$

Generally, in P^{3+1} two additional independent normal rotations-notions as if at a hyperbolic word line going with Poincaré 4-velocity **c** along a tangent $\mathbf{i}_\alpha$, with paired pseudonormal $\mathbf{p}_\alpha$, are mapping as rotation $d\alpha_1$ of $\mathbf{i}_\alpha$ at the local hyperboloid II in the sine normal plane and as rotation $d\alpha_2$ of $\mathbf{p}_\alpha$ at the local hyperboloid I in the cosine normal plane (with the concomitant movable trigonometric hyperboloids II and I of the radius-parameter 1 at any world line.)

Normal rotations $d\alpha$ do not change the hyperbolic angle of motion γ in its pseudoplane $P^{1+1} \equiv \langle \mathbf{i}_\alpha, \mathbf{p}_\alpha \rangle$ in (16), therefore they do not affect the proper time τ and do not increase the physical velocity v, but only change its Euclidean direction. In particular, this illustrates the Herglotz Principle in the kinematics and dynamics of TR. We can also map all 3 rotations in (23) at once onto the accompanying three-sheets quadrohyperboloid generated by the rotation of the quadrohyperbola around the arrow of time {ct}. All 3 accelerations in (23) are applied at the world point of object N in the tangent to the quadrohyperboloid $E^{3(m)} \subset P^{3+1}$ with its unit vectors $\mathbf{p}_\alpha$, $\mathbf{b}_\nu$, $\mathbf{b}_\mu$, giving Euclidean directions also for all 3 angular differentials. Rotations $d\alpha_1$ and $d\alpha_2$ are expressed in $E^{3(m)}$, and their sine and cosine hyperbolic projections are expressed in E_1 along $\mathbf{b}_\nu$ and $\mathbf{b}_\mu$ as if here with the action of *the hyperbolic Meusnier Theorem* in its sine and cosine variants on our three-sheets quadrohyperboloid.

As a result, we have a representation of all absolute characteristics in a pseudoorthogonal 4th-ortho reference frame in space-time P^{3+1}, which is *a movable tetrahedron* along an absolute Minkowski world line of object N:

$$\mathbf{p}_\alpha = \begin{bmatrix} \cosh\gamma_i \cdot \mathbf{e}_\alpha \\ \sinh\gamma_i \end{bmatrix}, \ \mathbf{b}_\mu = \begin{bmatrix} \mathbf{e}_\mu \\ 0 \end{bmatrix}, \ \mathbf{b}_\nu = \begin{bmatrix} \mathbf{e}_\nu \\ 0 \end{bmatrix}, \ \mathbf{i}_\alpha = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\alpha \\ \cosh\gamma_i \end{bmatrix}. \ \left(\mathbf{b}_\alpha = \begin{bmatrix} \mathbf{e}_\alpha \\ 0 \end{bmatrix}, \ \mathbf{i}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\mathcal{K}_\alpha = \eta^*_\gamma / c, \ \mathbf{k}_\alpha = \mathcal{K}_\alpha \, \mathbf{p}_\alpha; \ \mathcal{K}_\nu = \sinh\gamma_i \cdot w^*_{\alpha(1)}/c, \ \mathbf{k}_\nu = \mathcal{K}_\nu \, \mathbf{p}_\nu; \ \mathcal{Q}_\mu = \cosh\gamma_i \cdot w^*_{\alpha(2)}/c, \ \mathbf{q}_\mu = \mathcal{Q}_\mu \, \mathbf{p}_\mu.$$

In the differential motive tensor trigonometry, the term “motions” means increments of the motion angle. The motions are absent, iff the principal angle of the given trigonometry and geometry is zero. Due to pseudoorthogonality, $\mathbf{i}_\alpha$ and $\mathbf{p}_\alpha$, and due to Euclidean orthogonality $\mathbf{b}_\nu$ and $\mathbf{b}_\mu$, with also that all 4 basis vectors are orthonormalized in the tetrahedron, additionally 4D Minkowski space-time P^{3+1} can be represented as a direct sum of a pseudoplane $\langle \mathbf{i}_\alpha, \mathbf{p}_\alpha \rangle$ of hyperbolic motion $d\gamma$ and binormal Euclidean plane $\langle \mathbf{b}_\nu, \mathbf{b}_\mu \rangle$ with free *non-relativistic* axial rotation $d\alpha_3$ around the 4th binormal axis $\mathbf{b}_\alpha$ in $E^{3(k)}$ in the 1st decomposition (20), as the "Cardano gimbal" for the binormal plane in P^{3+1}:

$$\langle \mathcal{P}^{3+1} \rangle \equiv \langle \mathcal{P}^{1+1} \rangle_H^{(k)} \boxtimes \langle \mathcal{E}^2 \rangle_B^{(k)} \equiv \langle \mathbf{i}_\alpha, \mathbf{p}_\alpha \rangle_H^{(k)} \boxtimes \langle \mathbf{b}_\nu, \mathbf{b}_\mu \rangle_B^{(k)} \equiv \mathrm{CONST}. \tag{24}$$

In TR, quadratic representations of motions with the pair ‹$\mathbf{p}_\alpha$, $\mathbf{b}_\nu$› and with the pair ‹$\mathbf{i}_\alpha$, $\mathbf{b}_\mu$› are used – for details, see [10, pp. 271–274] and [10, pp. 276–278].

In *hyperbolic motions* $d\gamma_i$ (so, space flights with constant acceleration g_α) $\mathbf{e}_\alpha$=**const** determines the Euclidean direction of $d\gamma_i$ in $E^{3(1)}$. In *pseudoscrewed motions* $d\alpha_1$ (so, planetary in $E^{3(1)}$) angle γ_i =const gives the inclination of $d\alpha_1$ to the time-arrow {ct} as cosine and to $E^{2(1)}$ as sine.

Thus, we have in P^{1+1} the first simplest "hyperbolic motion" in *integral* form with 4-velocity of Poincaré along the tangent $\mathbf{i}_\alpha$ and in *differential* form with the curvature C_γ of the pseudo-Euclidean radius $R_\gamma=c^2/g$=const and with space-like acceleration $g_\alpha=c^2/R_\gamma$=const along the pseudonormal $\mathbf{p}_\alpha$, mapped in the basis E_1 on hyperboloids I and II with their pseudo-Euclidean radii R_γ=const [9], [10] here with tensor $\{I^{+-}\}$ [10, pp. 197–198]:

$$\left.\begin{array}{l} x = R_\gamma \cdot (\cosh\gamma - 1), \\ ct = R_\gamma \cdot \sinh\gamma. \end{array}\right\} \Rightarrow -(ct)^2 + (x+R_\gamma)^2 = R_\gamma^2 \cdot (-\sinh^2\gamma + \cosh^2\gamma) = +R_\gamma^2 \; [\mathrm{e}_\alpha = \mathbf{const},\ g_a = c \cdot (d\gamma/d\tau) = \mathrm{const}]$$

$$\left.\begin{cases} di\gamma \cdot \mathrm{p}_\alpha = \cos i\gamma \; di\gamma \cdot \mathrm{i}_1 + \sin i\gamma \; di\gamma \cdot \mathrm{b}_\alpha, \; (di\lambda = R_\gamma \, di\gamma) - \text{under Euclidean tensor } I^+. \\ C_\gamma^2 = 1/R_\gamma^2 = (d\gamma/dc\tau)^2 = \cosh^2\gamma \, (d\gamma/dc\tau)^2 - \sinh^2\gamma \, (d\gamma/dc\tau)^2 = \mathcal{Y}_\gamma^2 - \mathcal{X}_\gamma^2 > 0; \\ g_\alpha^2 = (c \, d\gamma/d\tau)^2 = c \, d\gamma/d\tau)^2 = (c \cdot \eta_\gamma^*)^2 = (c^* \eta_\gamma^*)^2 - (v^* \eta_\gamma^*)^2 = g_y^2 - g_x^2 > 0. \end{cases}\right\} \text{under } I^\pm \qquad (I)$$

Movable dihedron for motion $d\gamma$ includes time direction $\mathbf{i}_1$ and the binormal $\mathbf{b}_\alpha$, but simplest here movable *monohedron* includes only a pseudonormal $\mathbf{p}_\alpha$.

In complex Poincaré space-time with pseudo-spherical angle of motion $i\gamma$ [2], we can introduce as an alternative to it, but a *real-valued* orthospherical angle α_1 as if for the principal motion. Moving to realificated Minkowski space-time with its tensor $\{I^{+-}\}$, we convert their imaginary and real-valued nature into γ and $i\alpha_1$! This explains exactly how we obtain here very *strictly* the second simplest 3D "pseudoscrewed orthospherical motion" in integral form in TR Poincaré, again with the 4-velocity of Poincaré $\mathbf{c}$ along the principal tangent $\mathbf{i}_\alpha$ which is impotent here (i.e., without its curvature K_α) and in differential form with curvature C_R and radius R_C with unusual *time-like* (!) centripetal acceleration $g_\nu=c^2/R_C$ =const along *the* *normal tangent* $\mathbf{i}_\nu$, mapped to instantaneous center of pseudoscrewed motion O_M on the central axis – the time-arrow {ct} in E_1. A screw with pitch s is tangent to cylinder with axis {ct} and radius R=r (planetary physical motion), tangent to hyperboloid I along its equator with Euclidean radius R=r in the plane $E^{2(1)}$ – see in detail at [10, pp. 282–288]. The cylinder is the third accompanying at world lines trigonometric object of the radius parameter R and here in P^{2+1} (!).

$$\left.\begin{array}{l} r = R = R_C \cdot \sinh\gamma, \\ s = R_C \cdot \cosh\gamma > r, \end{array}\right\} \Rightarrow -s^2 + r^2 = R_C^2 \cdot (-\cosh^2\gamma + \sinh^2\gamma) = -R_C^2 \ [\gamma = \text{const}],\ \mathbf{i}_\nu = \begin{bmatrix} \sinh\gamma_i \cdot \mathbf{e}_\nu \\ \cosh\gamma_i \end{bmatrix}.$$

$$\left\{\begin{array}{l} d\alpha_1 \cdot \mathrm{i}_\nu = \cos i\gamma\ d\alpha_1 \cdot \mathrm{i}_1 + \sin i\gamma\ d\alpha_1 \cdot \mathrm{b}_\nu,\ (d\lambda = R_C\ d\alpha_1) - \text{under Euclidean tensor } I^+. \\ \mathcal{C}_R^2 = -1/R_C^2 = (di\alpha_1/dc\tau)^2 = \cosh^2\gamma\ (di\alpha_1/dc\tau)^2 - \sinh^2\gamma\ (di\alpha_1/dc\tau)^2 = -\mathcal{Y}_\alpha^2 + \mathcal{K}_\nu^2 < 0; \\ j_\nu^2 = (c\ di\alpha_1/d\tau)^2 = (c \cdot w_{\alpha_1}^*)^2 = (c^* w_{\alpha_1}^*)^2 - (v^* w_{\alpha_1}^*)^2 = -g_y^2 + g_\nu^2 < 0. \end{array}\right\} \text{under } I^\pm \qquad (II)$$

Movable trihedron for motion $d\alpha_1$ includes tangent $\mathbf{i}_\alpha$, binormals $\mathbf{b}_\nu$ and $\mathbf{b}_\alpha$, but simplest *movable dihedron* includes $\mathbf{i}_\nu$ and $\mathbf{b}_\alpha$. So, an impotent binormal $\mathbf{b}_\alpha$ is used for Euclidean direction of a *screw curve* as in the Frenet theory too in E^3 (!)

Pseudoscrewed motion with the Thomas precession around 3rd binormal $\mathbf{b}_\mu$ is described fully in в P^{3+1}. Trihedrons of screws in P^{2+1} and Q^{2+1} [10, p. 288–289] differ from Frenet trihedron in E^3. The point is that we use tensor trigonometry with its natural frame axis for definition of the principal angles. And therefore, the Frenet-Serret Theory in E^3 and our Theory in both binary spaces are distinct!

In elliptical screwed motion, the proportion between the space- and time-like differentials must observe, similar to $d\gamma_i$ in (I) and $di\alpha_1$ in (II). Then, in the sum of hyperbolic and pseudoscrewed motions, the total differential must have a time-like nature $(d\gamma_i)^2+(di\alpha_1)^2 < 0$. The planetary elliptical motion necessarily closes! When the sum is zero, the motion will be parabolic, as in Galilean geometry with its affine-Euclidean tensor trigonometry, see [10, pp. 168–171], which is intermediate between spherical and hyperbolic tensor trigonometry!

The time-like part of the curvature *Y* of a world line in P^{3+1} is referred to by us as "orthoprocession" (in E^3 for a closed curve or for a screw as the torsion).

Orthoprocession gives a permanent translational motion of object *N* with its world line parallel to the time-arrow {ct} at its supervelocity c*. When v<c, it stretches any world line along the axis {ct}: in system (I) $Y=\cosh\gamma_i\cdot[d\gamma/dc\tau]$ and system (II) $Y=\cosh\gamma_i\cdot[di\alpha_1/dc\tau]$. It is precisely because of orthoprocession in TR that cosine stretching of the proper time scale occurs (since $\cosh\gamma_i>1$) and the TR kinematic *Twin paradox of TR* is generated – see more details about this in Chapter 6.

Let us return to discussion of the contribution of orthospherical part to motion.

Orthospherical rotation of the normal part in (21) as $c\cdot[\sinh\gamma\cdot(d\alpha_1/d\tau)]\cdot\mathbf{b}_\nu$ and only together with the non-collinear parallel principal part cause in $E^{3(1)}\subset P^{3+1}$ and in TR the *orthospherical shift* around the 3rd normal axis $\mathbf{b}_\mu$. The integral shift Θ was expressed in discrete formulas (3)–(6), (8) and (19). Let us calculate it exactly and approximately, but now as the differential orthospherical shift –dθ. In formulas (4) and (5) for two-steps summation of hyperbolic motions – direct and inverse, we assume here: $\gamma_{12}=\gamma$ and $\gamma_{23}=d\gamma$. Then, from these, we find the Euclidean orthonormal vectors of the total non-collinear motion $e_\sigma=\{\cos\sigma_k\}$ – for the forward and backward order of summation, but with the desired shift. Then the vector product of these unit vectors in formula (8) is transformed into the vector differential $\{-d\theta\cdot\mathbf{e}_\mu\}$, i.e., here with factor (–dθ), see more details in [10, p. 226]:

$$\left.\begin{aligned}
&-\mathrm{d}\theta=-d\theta\cdot\overrightarrow{\mathrm{e}_\mu}=\mathrm{e}_{\underset{\sigma}{\angle}}\times\mathrm{e}_\sigma=\tanh\left(\gamma/2\right)\otimes\mathrm{d}\gamma=\frac{\tanh\gamma}{1+\operatorname{sech}\gamma}\otimes\mathrm{d}\gamma=\frac{\sinh\gamma}{\cosh\gamma+1}\otimes\mathrm{d}\gamma=\\
&=\frac{\sinh\gamma}{\cosh\gamma+1}\cdot\mathrm{e}_\alpha\otimes\left(d\gamma\cdot\mathrm{e}_\beta\right)=\frac{\cosh\gamma-1}{\sinh\gamma}\cdot\mathrm{e}_\alpha\otimes\left(d\gamma\cdot\mathrm{e}_\beta\right)=\\
&=\frac{\sinh\gamma}{\cosh\gamma+1}\cdot\sin\varepsilon\,d\gamma\cdot\overrightarrow{\mathrm{e}_\mu}=\frac{\cosh\gamma-1}{\sinh\gamma}\cdot\sin\varepsilon\,d\gamma\cdot\overrightarrow{\mathrm{e}_\mu}=\frac{\cosh\gamma-1}{\sinh\gamma}\,\overset{\perp}{d\gamma}\cdot\overrightarrow{\mathrm{e}_\mu}.\Rightarrow\\
&\frac{\mathrm{d}\theta}{d\tau}=w^*_\theta\cdot\overrightarrow{\mathrm{e}_N}=-\sin\varepsilon\cdot\tanh(\gamma/2)\cdot\frac{d\gamma}{d\tau}\cdot\overrightarrow{\mathrm{e}_N}=-\frac{\sin\varepsilon\cdot v^{(1)}\cdot g^{(m)}}{c^2\cdot\left(1+\operatorname{sech}\gamma\right)}\cdot\overrightarrow{\mathrm{e}_N}.\Rightarrow\\
&-d\theta/dt\approx(1/2)\sin\varepsilon\cdot g_\beta\cdot v/c=(1/2)\,\overset{\perp}{g}\cdot v/c=(1/2)\,w_\alpha\cdot(v/c)^2=(1/2)\tanh^2\gamma\cdot w_\alpha.
\end{aligned}\right\}\qquad(25)$$

We derived as if for a hyperbolic motion, but with a non-collinear hyperbolic differential increment, i.e., for the sum of the motion γ with its direction $\mathbf{e}_\alpha$ and dγ with its direction $\mathbf{e}_\beta$ or with its normal direction $\mathbf{e}_\nu$, trigonometric and derived directly from it physical formula *of differential* orthospherical shift in (3) in the base E_1, sometimes called "laboratory" one. We use the initial universal base E_1 as the base of relative rest in TO. For example, it can be seen that for small γ (velocities v), this precession is approximated by the area of a triangle with sides v/c and g/c and an angle ε between them. But this formula is greatly simplified for *orbital rotations with angular velocity* w_α. For this, we apply non-obvious useful *normal relations*. They follow from (21-I and II) for their Euclidean parts at $\gamma_p \neq 0$ and at $\gamma_p = 0$ in two forms of their expression *in* $E^{3(1)} \subset P^{3+1}$ [10, p. 240]:

$$\begin{cases} \gamma_p \neq 0 \to \cosh\gamma_p \cdot \sin\varepsilon\; d\gamma_p = \overset{\perp}{d\gamma_p} = \sinh\gamma_i\; d\alpha_1 \Rightarrow \cosh\gamma_p \cdot \sin\varepsilon \cdot g_\beta = \overset{\perp}{g_\beta} = v_i^* \cdot w_\alpha^*, \\ \gamma_p = 0 \to \sin\varepsilon\; d\gamma_p = \overset{\perp}{d\gamma_p} = \sinh\gamma_i\; d\alpha_1 \Rightarrow \sin\varepsilon \cdot g_\beta = \overset{\perp}{g_\beta} = v_i^* \cdot w_\alpha^*. \quad [\overset{\perp}{d\gamma_p}\ \&\ \overset{\perp}{g_\beta} \text{ are invariants in } (21, 22)!] \end{cases}$$

Let us continue transforming our general exact vector formula (25) for the differential orthospherical shift {–dθ·$\mathbf{e}_\mu$} at non-collinear and orbital motions, but taking into account the normal relations above:

$$\left.\begin{aligned} -\mathbf{d}\theta &= -d\theta \cdot \vec{\mathbf{e}_\mu} = \frac{\cosh\gamma - 1}{\sinh\gamma} \cdot \sin\varepsilon\; d\gamma \cdot \vec{\mathbf{e}_\mu} = \frac{\cosh\gamma - 1}{\sinh\gamma}\; \overset{\perp}{d\gamma} \cdot \vec{\mathbf{e}_\mu} = \\ &= (\cosh\gamma - 1)\; d\alpha \cdot \vec{\mathbf{e}_\mu} = k_E\; d\alpha \cdot \vec{\mathbf{e}_\mu} = [d\alpha^* - d\alpha] \cdot \vec{\mathbf{e}_\mu} \approx 1/2\; \gamma^2\; d\alpha \cdot \vec{\mathbf{e}_\mu}; \\ -\frac{\mathbf{d}\theta}{dt} &= w_\theta \cdot \vec{\mathbf{e}_\mu} = (\cosh\gamma - 1) \cdot w_\alpha \cdot \vec{\mathbf{e}_\mu} = k_E\; w_\alpha \cdot \vec{\mathbf{e}_\mu} = (w_\alpha^* - w_\alpha) \cdot \vec{\mathbf{e}_\mu}. \end{aligned}\right\} \quad (26)$$

Thanks to these normal relations, this formula extends through (25) to the pseudoscrewed motion, and physically to the orbital motions, as well as to the continuous non-collinear motions with α≠const! Moreover, in the pseudo-Euclidean space-time of TR and RCM, the same orthospherical shift is negative as (–dΘ) and (–dθ) in (25) and (26), but in the quasi-Euclidean binary space with q=1, n=3 or *2* in $E^{n(1)} \subset Q^{n+1}$, it is positive as +dΘ and +dθ and also relative to the sign of $d\alpha_1$ in the normal plane $E^{n(1)} \subset Q^{n+1}$ [10, p. 241] in quasi-Euclidean space. As a result, this orthospherical rotation with precession around the axis $\mathbf{e}_\mu$ is expressed most simply, concisely, and understandably *trigonometrically* as:

$$d\theta = d\alpha_1 - \cosh\gamma_i\; d\alpha_1 \approx -\gamma_i^2/2\; d\alpha_1 < 0 \quad \Leftrightarrow \quad d\theta = d\alpha_1 - \cos\varphi_i\; d\alpha_1 \approx +\varphi_i^2/2\; d\alpha_1 > 0. \quad (27, 28)$$

The point, that they are universal differential trigonometric formulas for all angular orthospherical deviations, such as Lambert's defect in the Lobachevsky hyperbolic geometry on upper part of Minkowski hyperboloid II, and Harriot's excess in Euler's spherical geometry, so, on our oriented hyperspheroid in Q^{3+1}. It generates the induced by $\gamma\neq0$ orthospherical rotations $\{-d\theta/d\tau\}$ in $E^{3(1)} \subset P^{3+1}$ with precession in time in enveloping P^{3+1} as the *Thomas precession* [15]. Thus, this process includes orthospherical orbital rotation of object w_α and induced axial negative rotation ($-w_\theta$) with the Thomas precession of 3rd axis $\mathbf{e}_\mu^{(1)}$ around the axis $\mathbf{e}_\mu^{(m)}$. Interestingly, this precession, as an energy rebound, also serves to fulfill the Law of Energy and Momenta Conservation in P^{3+1} [10, pp. 260, 264]. In GTR some relativists boast of alleged violations of the Law in a cosmic level. In TR on the Poincaré ideas, this Law is observed, even if only by applying Noether's Theorem in P^{3+1}. Thomas precession manifests itself when there is a motion Γ. As it is negative, it directly confirms physically the hyperbolic nature of the angle of motion! If there is no such motion, then there is no such precession. Therefore, it is *non-inertial* – see exact inference at formula (6)!

In RQM, in the orbital (formally spin) relativistic motion of a microobject, for example, an electron, the principal spin rotation moment and a small induced axial rotation moment arise with precession of the common 3rd normal axis $\mathbf{e}_\mu$ together with the plane of spin rotation of object in $E^{2(m)} \subset P^{3+1}$. Both of these moments have the same physical dimension, but they differ as spin and as a non-inertial moment of axial rotation with axis precession. Therefore, in RQM, they are split with the identification of the moment from the Thomas precession. It is expressed by us with the clear trigonometric formulas in logical chain (25)–(27) in the base E_1 with the approximated "half of Thomas" at $v \ll c$ and $\varepsilon = \pm\pi/2$. It should be noted that Thomas was the only person to date to receive the Nobel Prize for his work in the Theory of Relativity and with its experimental confirmation in RQM (1926), moreover in the original version from Poincaré [2] (1905) namely with the Lorentz transformation group.

In the base E_m, this rotation physically depends on acceleration and the angle between velocity and acceleration, as if "hidden" in the simplest formula (27):

$$w_\theta^* = \frac{d\theta}{d\tau} = w_\alpha^* - w_\alpha^* \cdot \sqrt{1+\sinh^2\gamma} = w_\alpha^* - w_\alpha^* \cdot \sqrt{1+\left[\frac{v^*}{c}\right]^2} = w_\alpha^* - \sqrt{[w_\alpha^*]^2 + \left[\frac{\sin\varepsilon\ g}{c}\right]^2} < 0. \qquad (29)$$

From (27), applying *the archaic relativistic factors of SRT* β and γ instead of trigonometric functions, we arrive at E_1 the long-known *physical formula* of Foppl and Daniel [16], who back in 1913 in Göttingen theoretically predicted this induced spherical precession as a kinematic effect of TR (possibly using the concept of "time dilation", introduced by Minkowski, also in the relation

$$w_\theta = d\theta/dt = -w_\alpha[1/\sqrt{1-\beta'^2} - 1] = -w_\alpha \cdot (\gamma' - 1).$$

After Henri Poincaré, Arnold Sommerfeld became the next pioneer in the application of trigonometry in the theory of relativity. In article [17], he derived the first formula from (25) in its scalar form by summing two segments on a hypothetical 2D Lambert sphere of imaginary radius iR, where γ and v → 0 resulted in "Thomas' half." Thus, we have finished considering orthospherical rotations — integral and differential in TR and in non-Euclidean geometries.

Chapter 5. Relativistic flight to the nearest star and its reality.

Let us apply our trigonometric approach to strictly derive the exact *relativistic* cosmic formula of Ziolkovsky for the hypothetical Sänger rocket [18], moving by the reactive force of photons [9, p. 234] (2004), [10, p. 203]:

$$F = m_0(\tau)\cdot g(\tau) = u\cdot\frac{dm_0(\tau)}{d\tau} \ \Rightarrow\ u\cdot\frac{dm_0(\tau)}{m_0(\tau)} = g(\tau)d\tau = c\,d\gamma(\tau) \Rightarrow$$

$$\Rightarrow m_0(\tau) = m_0\exp[-(c/u)\cdot\gamma(\tau)] = m_0\exp\{-(c/u)\cdot \operatorname{arsinh}\,[v^*(\tau)/c]\}, \qquad (30)$$

where m_0 and m are the initial and current masses of the rocket in the base E_m, and u is the fuel flow rate, γ(τ)=arsin[v*(τ)/c]. We are dealing here with the hyperbolic motion above in system (II). For hypothetical photon Sänger rocket (but as a theoretically ideal option), we obtain at u=c:

$$m_0(\tau) = m_0\exp[-\gamma(\tau)] = m_0\exp\{-\operatorname{arsinh}\,[v^*(\tau)/c]\} = m_0\exp\{-\operatorname{artanh}\,[v(t)/c]\}.$$

Let us compare the values of the rocket's own mass in terms of the coordinate and proper velocities of the photon rocket using trigonometric formulas above:

$$m_0 \exp(-v^*/c) < m_0 \exp[-\operatorname{arsinh}(v^*/c)] = m_0 \exp[-\operatorname{artanh}(v/c)] < m_0 \exp(-v/c). \quad (\sinh\gamma > \gamma > \tanh\gamma)$$

Let's say a hypothetical photon rocket flies to the star Proxima Centauri and its planets (the closest star system to us) and returns back to Earth.

The ideal parameters in terms of time for this flight are:

– fuel exhaust velocity u=c for a photon rocket (the theoretical maximum),

– constant acceleration g=10 m/sec^2, as on Earth, – coordinate time similar *to on a hyperbola* from t or its proper time on *a catenary* from τ ("chain line"),

– distance in one direction L=2x≈40.3.10^{15} m≈4.25 light years.

Let us consider trigonometric calculations for a round trip of reversible relativistic motion along a hyperbola with t and a catenary with τ.

For a such flight of the rocket, of course else hypothetical, we have trigonometric relativistic formulas:

$$\chi = L/2 = R\cdot(\cosh\gamma_{max} - 1), \quad \cosh\gamma = 1 + gx/c^2 = 1 + x/R \to (\cosh\gamma - 1) \sim x, \ (R = c^2/g);$$

$$\tau = 4(c/g)\gamma_{max}, \qquad t^{(1)} = 4(c/g)\sinh\gamma_{max}, \qquad t^{(1)}/\tau = \sinh\gamma_{max}/\gamma_{max};$$

$$v_{max} = c\cdot\tanh\gamma_{max}, \qquad v^*_{max} = c\cdot\sinh\gamma_{max};$$

$$m_0(\tau)/m_0 = \exp[4(-c/u)\gamma_{max}], \ \text{at } u = c: \ m_0(\tau)/m_0 = \exp[-4\gamma(\tau)], \ (\gamma = c\tau/R).$$

Our calculations gives the next numerical results:

$$\chi \approx 20.15\cdot 10^{15}\ \text{m}, \quad (L = 2\chi \approx 40.3\cdot 10^{15} m), \quad R \approx 9\cdot 10^{15}\ \text{m}, \quad t_F \approx 305\ \text{days};$$

$$\cosh\gamma_{max} \approx 3.239, \quad \sinh\gamma_{max} \approx 3.081 > 1, \quad \tanh\gamma_{max} \approx 0.951 < 1, \quad \gamma_{max} \approx 1.844$$

under action of this hyperbolic trigonometric inequality $\cosh\gamma > \sinh\gamma > \gamma > \tanh\gamma$;

$$v_{max} \approx 0.951c \ \text{ and } \ v^*_{max} \approx 3.061c \ \text{ with corresponding difference in both expenditures of time :}$$

$$t^{(1)} \approx 3.70\cdot 10^8\ \text{sec} \approx 11.7\ \text{years}, \ \tau \approx 2.21\cdot 10^8\ \text{sec} \approx 7,01\ \text{years} < 2L \approx 8,50\ \text{light years!}$$

The first estimates of relativistic space flights were made by Langevin [19].

This exact tensor trigonometric assessment of cosmic flight provides, in particular, a very clear and complete description of the "Twin paradox" in its ideal mode with Earth acceleration at the beginning and deceleration thereafter. As a result, we have that the 1st twin astronaut spends his own time τ≈7 years, and the 2nd twin stays on Earth until meeting brother t≈11.7 years at t/τ ≈1.67.

The Earth time t_c of light traveling there and back (for brother) at speed of "c", i.e. 2L≈8.50 light years, is *greater* than the proper time spent by the 1st twin astronaut! The superlight average velocity of the astronaut's cosmic flight is explained by the fact that his proper time was constantly reduced due to the action of inner acceleration g. But the light flew there and back at a speed of "c". In accelerated motion, it is possible to exceed the speed of light (!). Einstein's postulate in TR applies only to the tangent velocity $v<c$. Our heroic astronaut, with constant rocket acceleration g, flew there and back at an instantaneous sine velocity v*. The decrease in the rocket's own mass, according to the fuel consumption due to our relativistic formula, is equal to the unusual super-value: $m_0(\tau)/m_0=\exp(-4\gamma_{max})\approx 1/1600$! *This discredits such a flight with* g=10 *m/sec*^{2}.

We have shown above the equivalence of the reduction in time t/τ from the influence of the rocket's velocity v and from the influence of the rocket's acceleration g! Next, we will determine which of these two factors is primary and which is derived from the first. Thus, a photon rocket with Earth acceleration reaches its proper velocity v*=c in less than a year of the astronaut's proper time, and then its velocity increases to v*=3c. But at the end of its cosmic voyage, the rocket's own mass (even without cargo) will remain extremely small value ($m_0/1600$). Therefore, due to TR, such space flights, even to the nearest stars with the return of astronauts to Earth, are impossible for modern humans (but not for micro-robots with a significantly higher permissible g value). Although, seemingly serious physicists are putting forward much more absurd projects of space flights with the overcoming of supposedly curved space-time through tunnels called "wormholes" as another pseudoscientific populism.

The primary goal for ultra-long-range spaceflight in the foreseeable future is to accurately identify star systems capable of supporting Earth-like life, familiar to humans and our living world. Only then will it be possible to send a space expedition in their direction to explore new life in this part of the Universe.

Chapter 6. Cosmic flights in the Solar System and into deep cosmos.

The current 3-vector of the *proper* velocity $\mathbf{v}^*(\tau)=c\cdot\sinh\gamma\cdot\mathbf{e}_\alpha$ of the barycenter of object *N* is generated trigonometrically through integration with factor "c" of vector differential $d(\mathbf{sinh}\gamma)=d(\sinh\gamma\cdot\mathbf{e}_\alpha)$ in the proper time differential $d\tau$: $d(\sinh\gamma\cdot\mathbf{e}_\alpha)=\cosh\gamma_i\ d\gamma\cdot\mathbf{e}_\alpha+\sinh\gamma_i\ da_1\cdot\mathbf{e}_\nu=\cosh\gamma_p\cdot(\cos\varepsilon\ d\gamma_p\cdot\mathbf{e}_\alpha+\sin\varepsilon\ d\gamma_p\cdot\mathbf{e}_\nu)$ – see it above in decompositions (21-I and II) for the summary vector inner acceleration.

The proper velocity is the sine projection of Poincaré 4-velocity **c**, and physically as the result of continuous action of parallel and normal (to $\mathbf{e}_\alpha$) inner accelerations using the proper time τ:

$$\mathbf{v}^*(\tau)-\mathbf{v}^*(\tau_0)=c\cdot(\mathbf{sinh}\ \gamma-\mathbf{sinh}\ \gamma_0)=v^*(\tau)\cdot\mathbf{e}_\alpha(\tau)-v^*(\tau_0)\cdot\mathbf{e}_\alpha(\tau_0)= \quad (31)$$

$$=c\int_{\tau_0}^{\tau}\cos\varepsilon(\tau)\cdot\cosh\gamma_p(\tau)\cdot\frac{d\gamma_p}{d\tau}\,d\tau\cdot\mathbf{e}_\alpha(\tau)+c\int_{\tau_0}^{\tau}\sin\varepsilon(\tau)\cdot\cosh\gamma_p(\tau)\cdot\frac{d\gamma_p}{d\tau}\,d\tau\cdot\mathbf{e}_\nu(\tau)=$$

$$=\int_{\tau_0}^{\tau}\cosh\gamma(\tau)\cdot\left[c\cdot\frac{d\gamma}{d\tau}\right]\,d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}\left[c\cdot\sinh\gamma(\tau)\cdot\frac{d\alpha}{d\tau}\right]\,d\tau\cdot\mathbf{e}_\nu(\tau)=$$

$$=\int_{\tau_0}^{\tau}\frac{\overline{\overline{dv^*}}}{d\tau}\,d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}v^*(\tau)\cdot w_\alpha^*(\tau)\,d\tau\cdot\mathbf{e}_\nu(\tau)=$$

$$=\int_{\tau_0}^{\tau}\cosh\gamma(\tau)\cdot\overline{\overline{g}}(\tau)\,d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}\overset{\perp}{g}(\tau)\,d\tau\cdot\mathbf{e}_\nu(\tau),$$

where $d\alpha=d\alpha_1$ is the differential of 1st orthospherical rotations of vector $\mathbf{e}_\alpha(\tau)$;

$$\cosh\gamma\cdot\overline{\overline{g}}(\tau)=\frac{\overline{\overline{dv^*}}}{d\tau}=\overline{\overline{g}}^{\,*}(\tau),\quad c\frac{\overset{\perp}{d\gamma}}{d\tau}=\frac{\overset{\perp}{dv}^{(m)}}{d\tau}=\overset{\perp}{g}\,[t(\tau)]=v^*(\tau)\cdot w_\alpha^*(\tau).\quad (\overline{\overline{g}}^{\,*2}+\overset{\perp}{g}{}^2=g^2.)\quad (32,33)$$

At the proper velocity of object $v^*(\tau)=c\cdot[\sinh\gamma(\tau)]$ here appear its tangential acceleration and its normal inner acceleration – both in time τ, and $w_\alpha^*(\tau)=d\alpha/d\tau$ as proper angular velocity in the sine normal part of motion along a world line. The inner accelerations in (32, 33) generate the Relative and Absolute Pythagorean Theorems – see more about them in (21) and (22).

By analogy, the current 3-vector of the *coordinate* velocity $\mathbf{v}(t)=c\cdot\tanh\gamma\cdot\mathbf{e}_\alpha$ also is generated trigonometrically through integration with factor "c" of vector differential $d(\mathbf{tanh}\gamma)=d(\tanh\gamma\cdot\mathbf{e}_\alpha)$ in coordinate time differential dt.

The current 3-vector of the *coordinate* velocity **v**(t)=c·(tanhγ)·$\mathbf{e}_{(\alpha)}$ of the barycenter of an object or particle *N* is generated trigonometrically as the tangent projection of the Poincaré 4-velocity, and physically as parallel and normal accelerations using the coordinate time t:

$$\mathbf{v}(t) - \mathbf{v}(t_0) = c\cdot(\tanh\gamma - \tanh\gamma_0) = v(t)\cdot \mathrm{e}_\alpha(t) - v(t_0)\cdot \mathrm{e}_\alpha(t_0) = \tag{34}$$

$$= c\int_{t_0}^{t}\cos\varepsilon\cdot \mathrm{sech}^2\gamma_p(t)\cdot\frac{d\gamma_p}{dt}\,dt\cdot \mathrm{e}_\alpha(t) + c\int_{t_0}^{t}\sin\varepsilon\cdot \mathrm{sech}^2\gamma_p(t)\cdot\frac{d\gamma_p}{dt}\,dt\cdot \mathrm{e}_\nu(t) =$$

$$= \int_{\tau_0}^{\tau}\mathrm{sech}^2\gamma(\tau)\cdot\left[c\cdot\frac{d\gamma}{d\tau}\right]\,d\tau\cdot \mathrm{e}_\alpha(\tau) + \int_{\tau_0}^{\tau}\mathrm{sech}^2\gamma(\tau)\cdot\left[c\cdot\sinh\gamma(\tau)\cdot\frac{d\alpha}{d\tau}\right]d\tau\cdot \mathrm{e}_\nu(\tau) =$$

$$= \int_{t_0}^{t}\frac{d\overline{\overline{v}}}{dt}dt\cdot \mathrm{e}_\alpha(t) + \int_{t_0}^{t} v(t)\cdot w_\alpha^*[\tau(t)]\,dt\cdot \mathrm{e}_\nu(t) =$$

$$= \int_{t_0}^{t}\mathrm{sech}^3\gamma(t)\cdot\overline{\overline{g}}[\tau(t)]\,dt\cdot \mathrm{e}_\alpha(t) + \int_{t_0}^{t}\mathrm{sech}\,\gamma(t)\cdot\overset{\perp}{g}[\tau(t)]\,dt\cdot \mathrm{e}_\nu[\tau(t)],$$

where $t_0 = \tau_0,\ t = t(\tau)$.

Tensor Trigonometry also gives a simple and clear formulas for parallel and normal coordinate accelerations with formula for the inner force F initiating them and acting on object *N* at time t and τ:

$$\overline{\overline{g}}^{(1)}(t) = \mathrm{sech}^3\gamma\cdot\overline{\overline{g}}[\tau(t)] = \frac{d\overline{\overline{v}}}{dt},\quad \overset{\perp}{g}^{(1)}(t) = \mathrm{sech}\,\gamma\cdot\overset{\perp}{g}[\tau(t)] = \frac{d\overset{\perp}{v}}{dt} = v(t)\cdot w_\alpha^*[\tau(t)]. \tag{35,36}$$

$$\overline{\overline{F}} = \cos\varepsilon\cdot m_0 g = m_0\cdot\cosh^3\gamma\cdot\overline{\overline{g}}^{(1)}(t) \approx m\overline{\overline{g}},\ \ \overset{\perp}{F} = \sin\varepsilon\cdot m_0 g = m_0\cdot\cosh\gamma\cdot\overset{\perp}{g}^{(1)}(t) \approx m\overset{\perp}{g} = mvw\ (v \ll c). \tag{37}$$

These formulas (35)–(37), but up to now in a poorly understood physical form – using *archaic relativistic factors* of STR, instead of simplest and understandable trigonometric functions from the hyperbolic angle of motion, as we applicate, are quite widely used in relativistic physics, for example, in various physical particles accelerators.

The proper length of the way **x** of relativistic object or particle *N* is also estimated below in two variants with separation by time parameters $t_0=\tau_0$ and t=t(τ) under the condition of their simultaneity. In the basis E_1, from (31) and (34) we obtain two identical integrals for the same way **x** at the current τ<t and t:

$$\mathbf{x}_\tau(\tau)-\mathbf{x}_0\equiv\mathbf{x}_t(t)-\mathbf{x}_0=\int_{\tau_0}^{\tau}v^*(\tau)\cdot\mathbf{e}_\alpha(\tau)\;d\tau\equiv\int_{t_0}^{t}v(t)\cdot\mathbf{e}_\alpha(t)\;dt\equiv$$
$$\equiv\int_{\tau_0}^{\tau}\left[v_0^*\cdot\mathbf{e}_\alpha(\tau_0)+\int_{\tau_0}^{\tau}\cosh\gamma(\tau)\cdot\overline{\overline{g}}(\tau)\;d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}\overset{\perp}{g}(\tau)\;d\tau\cdot\mathbf{e}_\nu(\tau)\right]d\tau=$$
$$=\int_{\tau_0}^{\tau}\left[v_0^*\cdot\mathbf{e}_\alpha(\tau_0)+\int_{\tau_0}^{\tau}\overline{\overline{g}}^{\,*}(\tau)\;d\tau\cdot\mathbf{e}_\alpha(\tau)+\int_{\tau_0}^{\tau}\overset{\perp}{g}(\tau)\;d\tau\cdot\mathbf{e}_\nu(\tau)\right]d\tau\equiv$$
$$\equiv\int_{t_0}^{t}\left[v_0\cdot\mathbf{e}_\alpha(t_0)+\int_{t_0}^{t}\mathrm{sech}^3\gamma(t)\cdot\overline{\overline{g}}(t)\;dt\cdot\mathbf{e}_\alpha(t)+\int_{t_0}^{t}\mathrm{sech}\gamma(t)\cdot\overset{\perp}{g}(t)\;dt\cdot\mathbf{e}_\nu(t)\right]dt=$$
$$=\int_{t_0}^{t}\left[v_0\cdot\mathbf{e}_\alpha(t_0)+\int_{t_0}^{t}\overline{\overline{g}}^{(1)}(t)\;dt\cdot\mathbf{e}_\alpha(t)+\int_{t_0}^{t}\overset{\perp}{g}^{(1)}(t)\;dt\cdot\mathbf{e}_\nu(t)\right]dt. \qquad (38)$$

Variations of the time-like 1st differential of motion cosine are proportional to the work A of the tangential projection of the inner force F on direction $\mathbf{e}_\alpha$, containing such increments of motion, as in (11), but here with tensor $\{I^{-+}\}$ (!):

$$\left.\frac{d(ct)}{d(c\tau)}\right|_{\tau_0}^{\tau}=\int_{\gamma_0}^{\gamma}d\cosh\gamma=\int_{\gamma_0}^{\gamma}\sinh\gamma\;d\gamma=\int_{\gamma_0}^{\gamma}(\sinh\gamma\cdot\mathrm{e}_\alpha)\;(d\gamma\cdot\mathrm{e}_\beta)=\int_{\gamma_0}^{\gamma}\cos\varepsilon(\tau)\cdot\sinh\gamma\;d\gamma=$$
$$=\frac{1}{c^2}\cdot\int_{\tau_0}^{\tau}\cos\varepsilon(\tau)\cdot v^*(\tau)\cdot g(\tau)d\tau=\frac{1}{c^2}\cdot\int_{t_0}^{t}\cos\varepsilon[\tau(t)]\cdot v[\tau(t)]\cdot g[\tau(t)]\;dt=\frac{1}{c^2}\cdot\int_{\chi_0}^{\chi}\cos\varepsilon(\chi)\cdot g(\chi)\;d\chi=$$
$$=\frac{1}{m_0c^2}\cdot\int_{\chi_0}^{\chi}\cos\varepsilon(\chi)\cdot F(\chi)\;d\chi=\frac{1}{m_0c^2}\cdot\int_{\chi_0}^{\chi}\overline{\overline{F}}(\chi)\;d\chi=\frac{A}{m_0c^2}=\frac{A}{E_0}=\frac{\Delta E}{E_0}=\cosh\gamma-\cosh\gamma_0. \qquad (39)$$

At $\gamma_0=0$: $\boxed{A/m_0c^2=A/E_0=k_E=\cosh\gamma-1}\Rightarrow\boxed{E=\cosh\gamma\cdot E_0=m_0c^2+A=E_0+A=mc^2}$.

The scalar $E_0=m_0c^2$ and vector $\mathbf{P}_0=m_0\mathbf{c}$ formulas for energy and 4x1-momentum of object *N* on a world line are validated from (10) and (11) as trigonometric 4x4-tensor of energy-momentum and 4x1-momentum. For the first time, as *scalar*, these formulas for energy and momentum were applied in the pioneer article [20, p. 260] (1900) by the great and universal scientist and philosopher of the Science Henri Poincaré for electromagnetic radiation of light in two specific forms: mc=p=E/c and F=mc/t=E/ct=N/c. It is important that Poincaré discovered the inertia of radiation and its mass by estimating the enormous *energy* E and *power* N corresponding to only a gram of radiation. What is essential in his energy and momentum balances is that in Maxwell's electromagnetic theory of light there is parameter **П**/c2, where **П** – is the Poynting vector and $c^2=1/(\varepsilon_0\mu_0)$ was *an experimental coefficient* that arose in a rational system of physical units with the unit of current "Ampere" instead of the CGS system of units of Gauss!

A similar situation with "c^2" exists in gravity, where it appears after the translation from dimensionless tensor trigonometry to physical potentials — gravitational and accelerational. Then, the factor $c^2=1/(\varepsilon_0\mu_0)=E/m=(\nu\lambda)^2$ unites TR, electromagnetism, and gravity, but only in P^{3+1}, and in Q_c^{3+1} (!).

Later this energy-mass formula was obtained as $m=E/c^2$ by Einstein in 1905 for the mass of electromagnetic energy of thermal radiation in [21] and as $E=mc^2$ by Lewis in 1908 for the kinematic energy of relativistic motion in [22]. The priority of its discovery belongs to Poincaré from 1900. In reality, the formula is revealed precisely in P^{3+1}. In GTR, this formula was taken from STR to replace mass, but in GTR it is not derived! Then, where is the logic of GTR?

For travel into deep cosmos, we must assume that the normal vector $\mathbf{e}_\nu$ and $\mathbf{e}_\mu$ must absent in the relativistic formulas (31)–(39), and $\mathbf{e}_\beta=\mathbf{e}_\alpha$ at motion along $\mathbf{e}_\alpha$.

Глава 7. Ускорительный и гравитационный косинусы, влияние на время.

From (11) and (16)–(19) it follows that the time dilation $dt/d\tau$ and the energy $E=mc^2$ of object N is controlled by the hyperbolic cosine $\cosh\gamma=d(ct)/d(c\tau)$ from the angle of motion γ at the world line of object N through the angular scalar elements 4x4-tensors in (11) with them. But in the 3x3 Euclidean part of the tensors, cosine participates only in the evaluation of work $A=\Delta E$ in projection onto the Euclidean direction of motion $\mathbf{e}_\alpha$. It follows that in the relativistic motion of N, the cosine relations for time dilation $dt/d\tau$ and energy addition E/Eo or potential addition Π/Π_0 are equal. This is even more obvious due to the proportionality of pseudo-Euclidean invariants in (11) with the common for them angle of motion γ at a world line of a massive object N in the Minkowsspace-time P^{3+1} with the tensor I^{-+} with the *positive* angular cell (+1), as in (39), making both energy E and potentials Π here also positive (!) at the time arrow {ct}, but now with *an imaginary E^3* (!) – as in the GTR with its curve E^3. Let us introduce the new concept of *energy ratio*, which is important for further:

$$k_E=\cosh\gamma-1=\frac{d(ct)}{d(c\tau)}-1=\frac{E}{E_0}-1=\frac{\Delta E}{E_0}=\frac{A}{E_0}=\frac{m_0\cdot\Delta\Pi}{m_0\cdot c^2}=\frac{\Delta\Pi}{c^2}=\frac{\Delta\Pi}{\Pi_0}=\frac{\Pi-\Pi_0}{\Pi_0}=\frac{\Pi}{\Pi_0}-1. \quad (40)$$

The positive relative scalar potentials $\Delta\Pi_j$ with tensor $\{I^{-+}\}$ from various causes, but at the same world point, are summed additively (with the dimension of the square of velocity). When all these $\Delta\Pi_j$ are equal, then we obtain the simplest additive – exact and approximate formulas for their summation at the given world point of object *N* on its world line:

$$\frac{d(ct)}{d(c\tau)} - 1 = q \cdot k_E = q\frac{\Delta\Pi}{\Pi_0} = q\frac{\Delta\Pi}{c^2} = q \cdot (\cosh\gamma - 1) \approx \cosh^q\gamma - 1 \quad (\text{ the latter at } v \ll c, \text{ or } \gamma \to 0). \quad (41)$$

Let us establish a connection between any single relative potential $\Delta\Pi$ and the achieved sine or tangent 3-velocities, i.e., *integrally* or as in *discrete* Lorentz transformations from E_1 to E_m in their canonical form, for example, in (1) and below as if in the absence of gravity:

$$\cosh\gamma = \frac{d(ct)}{d(c\tau)} = 1 + \frac{\Delta\Pi}{c^2} = \sqrt{1+\sinh^2\gamma} = \frac{1}{\text{sech}\gamma} = \frac{1}{\sqrt{1-\tanh^2\gamma}} = \sqrt{1+(v^*/c)^2} = \frac{1}{\sqrt{1-(v/c)^2}} \to$$

$$\to \Delta\Pi_a = k_E \cdot c^2 = (\cosh\gamma - 1)\cdot c^2 = [\sqrt{1+(v^*/c)^2} - 1]\cdot c^2 = \frac{c^2}{\sqrt{1-(v/c)^2}} - c^2 \approx \frac{v^{*2}}{2} \approx \frac{v^2}{2}. \quad (42)$$

By expanding the hyperbolic cosine of γ in a series, we express the same representations discretely, but by a simpler and more universal way:

$$\Delta\Pi = k_E \cdot c^2 = (\cosh\gamma - 1)\cdot c^2 \approx \frac{\gamma^2}{2}\cdot c^2 \approx \frac{v^{*2}}{2} \approx \frac{v^2}{2}, \quad (\sinh\gamma > \gamma > \tanh\gamma). \quad (43)$$

In the case of free motion of object or particle N caused by the gravitational action of an astronomical mass M, equivalence of the gravitational potential Π_f and the generated kinematic potential Π_a is true at any world point of N:

$$\frac{fMm}{R} = \Delta E \approx \frac{mv^{*2}}{2} \approx \frac{mv^2}{2} \;\Rightarrow\; \frac{fM}{R} \approx \frac{v^{*2}}{2} \approx \frac{v^2}{2} \;\Rightarrow\; \Delta\Pi_f = \frac{fM}{R} = \frac{\Delta E}{m} = \Delta\Pi_a > 0. \quad (44)$$

Let us introduce a 4x4 tensor $T_\Pi = c^2 \text{roth}\Gamma$ of potential *N* in (9)–(11), but now with tensor $\{I^{-+}\}$ as above. We will be interested in its *additive angular cell* Π:

$\Pi = \Pi_0 + \Delta\Pi = c^2 + \Delta\Pi$, in particular, proportional to this cell in $\mathbf{T}_E$ in (11):

$E = E_0 + \Delta E = m_0c^2 + \Delta E$, although other cells are interesting, but for other goals.

In Minkowski space-time P^{3+1}, in cosmic region with astronomical mass M, its *Newtonian gravitational potential*, mathematically related to the angular cell of the tensor $\mathbf{T}_{\Pi}$ above, *has a* (+) *sign under the action of the alternating unity metric tensor* $\{I^{-+}\}$, when its corner cell (+1) corresponds to the realificated time-arrow {ct}. Then Π_0, $\Delta\Pi_f$, $\Delta\Pi_a$ will be *positive, but time-like concepts* (such as the concepts Π and E in GTR). But in the complex quasi-Euclidean space-time of Poincaré Q_c^{3+1} with *a Euclidean metric tensor* $\{I^{+}\}$ and with *a reflector tensor* $\{I^{+-}\}$ and an imaginary time arrow {ict}, *energy* and *potentials are negative and time-like*, as is customary *only for* Π *in classical physics* (?!).

In TR, discrete Lorentz transformations for the coordinates of object *N* are applied when they are translated from the initial base of relative rest $E_1 = \{I\}$ to the base of motion E_2 or E_m in P^{3+1}. But in TR, the Galilean base E_m can be *instantaneous in non-inertial motion* with acceleration and deceleration! Their simplest and most understandable canonical tensor trigonometric forms in E_1 in (1) and (2) is theoretically conditioned by the homogeneity and isotropy of Minkowski space-time. In *continuous* accelerated motion, with the rest base E_1 and instantaneous bases E_m, namely (see, for example, in [10, p. 236]) *the internal acceleration of object N has an effect on the local dilation of proper time* dτ relative to coordinate time dt in E_1 as

$$g = c\, d\gamma/d\tau \rightarrow d\tau = c\, d\gamma/g \quad (\gamma = \operatorname{arsinh} v^*/c = \operatorname{artanh} v/c) \tag{45}$$

along the way **x** in E_1, since the angle of motion γ is expressed in the initial E_1.

In Minkowski space-time P^{3+1}, in the presence of gravity, the inner kinematic acceleration g_a and the local gravitational intensity g_f influence differentially and integrally equivalently on the relativistic dilation of proper time τ of a moving object with its Newtonian also equivalent kinematic and gravitational mass [23].

In the applications of Tensor Trigonometry to Relativistic Physics, *the energy ratio* $k_E = (\cosh\gamma - 1)$ acts either with energy addition as in (41) and as in the shift of the perihelion of Mercury's orbit (see below) or with energy rebound as in the Thomas precession in (27). With Newtonian equivalence of the concepts of mechanical (kinematic) and gravitational accelerations, we further introduce the trigonometric concepts of acceleration and gravitational cosines [10, p. 259], and also reveal strictly the relationship k_E with the relative potential ΔΠ for any material object *N* moving from the action of the tangential acceleration $\mathbf{g}_\alpha$:

$$d\cosh\gamma = \sinh\gamma\ d\gamma = \frac{v^*}{c}\cdot\frac{g}{c}\,d\tau = \frac{g\ dx}{c^2} = \frac{F\ dx}{m_0c^2} = \frac{dA}{m_0c^2} = \frac{dE}{m_0c^2} = \frac{|d\Pi|}{c^2}. \quad (46)$$

For rectilinear motions we have *increasing* time dilation with final factor k_E:

$$\cosh\gamma_{(a)} = \frac{d(ct)}{d(c\tau)} = 1+\int_0^x \frac{g_a\ dx}{c^2} = 1+\int_0^x \frac{F_a\ dx}{m_0c^2} = 1+\int_{\Pi_0}^{\Pi_a} \frac{d\Pi_a}{c^2} = 1+\frac{\Delta\Pi_a}{c^2} = 1+\frac{A}{E_0} \to k_E = \frac{\Delta\Pi_a}{c^2}. \quad (47)$$

$$\cosh\gamma_{(f)} = \frac{d(ct)}{d(c\tau)} = 1+\int_0^x \frac{g_f\ dx}{c^2} = 1+\int_{\Pi_0}^{\Pi_f} \frac{d\Pi_f}{c^2} = 1+\frac{\Delta\Pi_f}{c^2} = 1+\frac{fM_0}{r\cdot c^2} \to k_E = \frac{\Delta\Pi_f}{c^2}. \quad (48)$$

For circular normally accelerated motions, we have *constant* time dilation with its single factor k_E:

$$\cosh\gamma_{(a)} = \frac{d(ct)}{d(c\tau)} = 1+\frac{\Delta\Pi_a}{c^2} = 1+\frac{A}{m_0c^2} = 1+\frac{\Delta E_a}{m_0c^2} \approx 1+\frac{v^{*2}}{2c^2} = 1+\frac{v^*w_r^*\cdot r}{2c^2} = 1+\frac{g_a\cdot r}{2c^2} = 1+\frac{(r\cdot w_r^*)^2}{2c^2}. \quad (49)$$

$$\cosh\gamma_{(f)} = \frac{d(ct)}{d(c\tau)} = 1+\frac{\Delta\Pi_f}{c^2} = 1+\frac{fM_0}{r\cdot c^2} = 1+\frac{\Delta E_f}{m_0c^2} \approx 1+\frac{v^{*2}}{2c^2} = 1+\frac{v^*w_f^*\cdot r}{2c^2} = 1+\frac{g_f\cdot r}{2c^2} = 1+\frac{(r\cdot w_r^*)^2}{2c^2}. \quad (50)$$

Thus, the acceleration cosine in the base E_1 bonds with acceleration and potential by formulas (47) and (48); the gravitational cosine in the base E_1 bonds with intensity and potential by formulas (49) and (50).

Let us assume logically that $\Pi_0=c^2$ is *the reference potential of the Universe*. Then $\Delta\Pi=\Pi-\Pi_0$ is the relative potential and is considered relative to Π_0, which appeared earlier in (40). And ΔΠ is not defined by an absolute manner, just like the relative velocity v. It does not depend on the mass of an object *N*.

For example, in the solar system, with the mass M of the Sun, its potential characteristic can be determined at a distance R from the center of the Sun either through the 1st cosmic velocity as $v^2\{I\}=fM/R$, or through the 2nd cosmic velocity as double $v^2\{II\}=2fM/R$, where cosmic velocity is greater than the first by √2 times; – or identically as $\Delta\Pi_f=v^2\{II\}/2=v^2\{I\}=fM/R$.

If we artificially equate both of these two velocities to "c", then for a spherical "black hole" its conventional radii of Mitchell (1783) and Schwarzschild (1916) are also derived from the 1st and 2nd cosmic velocities "c", so, for photons. These radii are *vague* concepts and cannot be measured. The coefficient "2", taking into account the events location in P^{3+1}, can be explained easily by the equivalence of accelerational and gravitational potentials for free motions with the summation of two equal Π at the events horizon of the "black hole". Both of these potentials are created here by centripetal acceleration g_a and tension g_f in formulas (49) and (50).

In the most general case, for free relativistic motion of any mass m under the action of gravity, with equivalent actions on its proper time of accelerational and gravitational cosines through acceleration and gravity potentials in (47), (48) or (49), (50), we have from (41) the doubled values of the relative potential ΔΠ, for example, with its approximation to $v^{*2} = v^{*2}\{II\}$ or to $v^2 = v^2\{II\}$ (i.e., also as for the 2nd cosmic velocity) due to the equivalence of the inner acceleration g_a and the gravity tension g_f. With the potential increasing by the factor of "2", the hyperbolic angle of motion γ_σ and its cosine increase at q=2, as indicated below:

$$\cosh\gamma_\sigma = \frac{d(ct)}{d(c\tau)} = 1 + 2\frac{\Delta\Pi}{c^2} = \frac{c^2 + 2\Delta\Pi}{c^2} = \frac{\Pi_0 + 2\Delta\Pi}{\Pi_0} = \frac{\Pi_0 + \Delta\Pi_\sigma}{\Pi_0} = \frac{\Pi}{c^2} = \frac{\Pi}{\Pi_0} \approx \cosh^2\gamma > \cosh\gamma, \quad (51)$$

$$\cosh\gamma_\sigma - 1 = 2\frac{\Delta\Pi}{c^2} = \frac{\Delta\Pi_\sigma}{c^2} = \frac{\Delta\Pi_\sigma}{\Pi_0} = 2\cdot k_E = 2\cdot(\cosh\gamma - 1) \approx \cosh^2\gamma - 1 \quad (\text{at } v \ll c). \qquad (52)$$

If we theoretically move to the world line of object *N* in P^{3+1}, where the absolute 4-velocity of Poincaré **c** acts, then there this object has an absolute potential $\Pi = \Pi_0 + \Delta\Pi_\sigma$ (under the tensor $\{I^{-+}\}$) acting generally in the absolute motion of matter, – in particular, including a proportional influence of the ratio $\Pi/\Pi_0 = \cosh\gamma > 1$ on the flow of proper time dτ dilation for object *N*:

$$\cosh\gamma_\sigma = \frac{d(ct)}{d(c\tau)} = \frac{E}{E_0} = \frac{mc^2}{m_0c^2} = \frac{m_0\Pi}{m_0\Pi_0} = \frac{\Pi}{\Pi_0} = 1 + \frac{\Delta\Pi_\sigma}{\Pi_0} = 1 + \frac{\Delta\Pi_\sigma}{c^2} = \frac{c^2 + \Delta\Pi_\sigma}{c^2} = \frac{\Pi_0 + \Delta\Pi_\sigma}{\Pi_0}. \quad (53)$$

The fundamental relationship $E_0=m_0c^2$ and $\Pi_0=c^2$ seem to be separate entities, but they must obviously be related. Modern relativists use almost exclusively the variant $E=mc^2$ and it's a pity. The concepts E_0 and m_0 for a physical object at rest are also fundamental and interesting characteristics. But first, let us return to *the relationship* $\Pi_0=c^2$, since in it the speed of light "c" is functionally related to the value *of the reference potential of the Universe* Π_0 – of the same in various cosine formulas (47)–(50) for the time dilations from equivalent kinematic acceleration and gravity tension. If hypothetically this reference potential Π_0 is not a constant value, then "c" also should not be an absolute constant in the Universe. It should be determined by the genesis of the reference potential Π_0.

In order to answer the cardinal and actual question of our time in the Science: "*Is the speed of light* "c" *in a cosmic vacuum a strictly constant value or not*?", – we first need to clear out the essence of the Universe reference potential $\Pi_0=c^2$ in (40) and (53) – *positive* with the accepted metric tensor $\{I^{-+}\}$ as also in GTR (for the positivity of energy and potential) and with all positive corner cells. Then potential $\Pi_0=c^2$ must have some kind of understandable natural origin with physical meaning in the space-time P^{3+1} and wider in the Nature.

"We logically formulate that at any point in the Universe Π_0 *is produced by the sum of the relative gravitational potentials* $\Delta\Pi_{f(j)}$ *of all material objects and other matter in it, if their number is finite, or by their convergent integral sum, if it is infinite.*" (And the relative potentials of matter motion $\Delta\Pi a_{(k)}$ are included.)

It follows from this statement that the Universe reference potential Π_0 (with tensor $\{I^{-+}\}$) determines the speed of light "c" in cosmic vacuum and the scale factor "c" of Poincaré at the time-arrow {ct} at a given place and a given time. That is, the speed of light in a vacuum is not a constant, but primarily a function $c=F(\Pi_0)=\sqrt{\Pi_0}$, where Π_0 is functionally strictly tied to the place **x** and the time t, or to the 4x1 world point **u**, for example, in the instantaneous base E_m applied in it, as *a response function* from **u**. In other words, Π_0 depends on the distribution of matter in the Universe relative to any world point **u**!

Given that the absolute potential is $\Pi=\Pi_0+\Delta\Pi_\sigma$, then the cosine of the hyperbolic angle at the world line of object N at its world point is equal to $\cosh\gamma_\sigma=(\Pi_0+\Delta\Pi_\sigma)/\Pi_0=1+\Delta\Pi_\sigma/\Pi_0\geq1$. It is not limited from above starting from 1.

Despite the relative proximity of the Sun to us on the Earth, its one-time potential contribution $\Delta\Pi_f$ to the value "c" is very small namely by this reason. Thus, in the vicinity of the Sun's surface we have $R=6.95\cdot10^8$ m, and hence it is: $\Delta c=[\sqrt{(c^2+\Delta\Pi_{f(s)})}-c]=[\sqrt{(c^2+v^2\{I\}_s)}-c]=c[\cosh\gamma-1]=c\cdot k_E\approx+317$ m/sec.

Such deviations "c" in the solar system are insignificant, but, in principle, they can be determined, for example, by radar observation of the Mercury.

However, the Sun acts with a directed intensity g_f of its gravity, which leads to all GR-effects in the solar system. From the point of view of the Universe as a whole, our Sun is only a microscopic object – a lost ant on the edge of one of the numerous small spiral galaxies Milky Way. (How do our people look in them?)

Therefore, physicists assume the local speed of light to be constant in all formulas and equations involving it – even for the entire or infinite Universe as the "world constant c." In Maxwell's Electromagnetic Theory of Light (1873), the same initial factor c^2 was a certain experimental coefficient in the system of measures with the unit of current "Ampere" in comparison with Gauss's CGS. And it turned out to be surprisingly close to the squared light speed, which had already been quite accurately estimated by astronomer James Bradley (1728).

This gives us a direct connection in the united formula $c^2=\Pi_0=1/(\varepsilon_0\mu_0)=(\nu\lambda)^2$ electromagnetism, gravity in TR with RQM, as well as the fact that photons with their electromagnetic mass are deflected by gravity normal to its path projection of the intensity $\mathbf{g}_f$. Then now we can return to interpretation of E_0 and m_0 with E.

1. If potential $\Delta\Pi_f=f\cdot M/R$ acts on mass m_0, then this gives to it "energy excess" $\Delta E=\Delta\Pi_f\cdot m_0=f\cdot M\cdot m_0/R\approx m_0v^2\{II\}/2=mv^2\{I\}$, as in (44).

2. If potential Π_0 acts on mass m_0, then this gives to it "own energy" $E_0=\Pi_0\cdot m_0=m_0c^2$, as in (39)!

3. If Π_0 acts on m, then this gives to it "total energy" $E=\Pi_0\cdot m=mc^2$, as in (53)!

This understandably and simply explain the physical origin of conceptions own and total energies of any material object and of the matter in our Universe!

From this at all, in addition, we conclude that m_0 and m are simultaneously *inertial*, *gravitational* and *electromagnetic relativistic mass* – own and total!

From the point 1 above we may conclude that "c" is also 1st cosmic velocity for our Universe as a whole, that is logically intuitively!

Let us also discuss the possible history of Π_0 genesis. So, an affine topology of the space-time of Nature and, as its good enough model, P^{3+1} gives it the properties of unboundedness and infinity. However, an infinite space-like 3D Euclidean part of our 4D world with uniformly distributed matter must have, according to the Olbers' Paradox (1826), a bright night sky – as opposed to the finite world of the Universe with radius-parameter R. But mathematical infinity of P^{3+1} does not at all imply the infinity of the mass of all world matter and, accordingly, its uniform distribution in the real Universe. These assumptions may not actually exist. A priori, the geometry of real space-time in the large is not discussed here, and it is not known to anybody!

Complete knowledge of the global structure of our universe is, in principle, unattainable. Scientists' previous illusions about achieving complete knowledge in mathematics were destroyed by Gödel's Incompleteness Theorems. But in the theoretical physics, the idea of the transcendence of the nature of the Universe as a whole is still far from being realized. Currently, there are two main hypotheses for the origin and development of the material world of the Universe.

Either, according to George Gamow's Big Bang theory, when all the mass of matter appears; or, according to Roger Penrose's Pulsating Universe theory, where the mass and energy of matter are conserved. If any is proven, then the light speed and Einstein's photons with Poincaré scale factor in his time-arrow can change significantly in the critical phases of the creation of the potential Π_0. Changes in the light speed are also possible at the supermassive cosmic objects.

Next, we need to justify that our approach to the Theory of Relativity with the gravity using accelerational and gravitational cosines and potentials accurately corresponds, first of all, to known astronomical deviations in the solar system!

Chapter 7. Trigonometric cosine and potential interpretations of GR-effects.

For a rigorous interpretation of all GR-effects in the solar system within the framework of TR in Minkowski space-time P^{3+1}, we apply the accelerational and gravitational cosines and potentials introduced above. We derive accurate clear trigonometric formulas for all GR-effects with also clear physical interpretation, following the approaches in the 3rd edition of our Tensor Trigonometry [10]. However, apologists for the General Theory of Relativity treat their approximate physical formulas for all GR-effects as consequences of GTR without proof and as peremptory. The approximate character of GTR formulas for these GR-effects is a direct consequence of the curvature of space-time! But this acts many wider.

The average reader, who is not well versed in the very complex absolute tensor calculus, must trust only GTR numerous mysterious formulations such as "This is a consequence from the equations of GTR" without any clear and understandable specific physical explanation. This creates the appearance that it is something inaccessible to the average person, but only to the elite from the seemingly closed grant-holder Club of GTR. We achieve correct results and not approximate, but mathematically and physically strict and accurate, by applying a much more simple and intuitively understandable *orthogonal tensor calculus* from Tensor Trigonometry with using Minkowski space-time as natural for TR.

At the same time, we return the Theory of Relativity to its historically original form by Henri Poincaré with the Lorentz transformations, Minkowski world lines, Poincaré absolute 4-velocity for any type of matter, and the Lobachevsky hyperspace and its geometry for a representation of all relativistic kinematics (velocities and accelerations) and dynamics (energy and all momenta), and much, much more that was simply taken away from TR with the advent of GTR.

Apologists for GTR have been writing and propagating for over 100 years that the GR-effect "Relativistic shift of Mercury perihelion" can only be explained within the framework of GTR, and they categorically assert that this effect confirms the truety of GTR. Our trigonometric solution with its physical interpretation within the framework of Newtonian theories and TR in P^{3+1} is based on three cosine dilations of proper time for Mercury with their doubling due to the equivalence of the acceleration and gravitational cosines during its free motion from the action of the Sun gravity on it. It is originally based on the application of *pseudo-Euclidean* Tensor Trigonometry methods in [9] and [10].

If Mercury passed Newton's orbit at velocity **v** in E_1, then there would be no time dilation for it. But Mercury passes it at velocity **v*** in its proper time τ in E_m (as in the Twin Paradox). Its real orbit was formed under relativistic parameters. The spent proper time is reduced according to the formula with one "k_E" as: $\delta=L/(dx/dt)-L/(dx/d\tau)=(\cosh\gamma-1)\cdot L/v^*=k_E\cdot L/v^*$. The number of the Mercury revolutions around the Sun in equivalent complete periods in time does not differ in the bases E_1 and E_m! Because of this, the complete orbit from perihelion to perihelion shifts forward with Mercury perihelion to compensate for the difference in time of its passage in the bases E_1 and E_m in P^{3+1}. The noticeable eccentricity of its orbit allowed the famous astronomer Le Verrier in 1859 to discover "on tip of a pen" (to Arago) this small effect, additional to the non-relativistic effects of the perihelion shift from the influence of other planets in the solar system. And to evaluate this GR-effect quite accurately, we adopt below:

(1) The motion of the planet Mercury in its orbit is almost circular.

(2) In the circular potential formulas (49) and (50), we use the exact value of Mercury's kinematic potential from (43) $\Delta\Pi_a=(\cosh\gamma-1)\cdot c^2$ and, from (44), its relativistic approximation $\Delta\Pi_a=\Delta\Pi_f\approx v^{*2}/2=fM/R$ – all again with tensor $\{I^{-+}\}$. Note that point (2) corresponds to the transition to proper time τ in Mercury on its orbit with preservation of Lorentz invariance in Minkowski space-time P^{3+1}.

But in the so-called "Schwarzschild solution" within the framework of general relativity, the transition to proper time τ meant the loss of Lorentz invariance!

Additionally, for the so-called "normal mass" of Mercury in orbit, acting in Newtonian Law of gravitation, in fact, due to the absence of motion here along of the orbit radius between the center of the Sun and the center of the Mercury, we apply there the proper mass of Mercury as m_0 while maintaining its total parallel mass m along its orbit. There are three equal relativistic cosine factors coshγ that dilate Mercury proper time: two v*/v and one m/m_0 compared to only one factor in (49) and (50). They get doubling these dilations of Mercury time due to equivalence of the accelerational and gravitational cosines in free motion as $6k_E=6\cdot(\cosh\gamma-1)$. Using (41), finally we obtain an enough exact estimate of this GR-effect in (54) and its approximate formula by Gerber [24], highlighted in (55), with a six-fold cosine time dilation from six factors $k_E>0$ [10, p. 263]:

$$\delta = +T\cdot 6\,k_E\cdot\frac{d\alpha}{dt} = +T\cdot 6\,(\cosh\gamma-1)\cdot w_\alpha = \frac{6\cdot 2\pi R}{v}\cdot(w_\alpha^* - w_\alpha) = 6\pi\cdot 2(\cosh\gamma-1) = 6\pi\cdot(\cosh\gamma_\sigma - 1) = 6\pi\cdot\frac{\Delta\Pi_\sigma}{c^2}. \quad (54)$$

$$\delta = +T\cdot 6\,k_E\cdot\frac{d\alpha}{dt} = +T\cdot 6\,(\cosh\gamma-1)\cdot w_\alpha = \frac{6\cdot 2\pi R}{v}\cdot(w_\alpha^* - w_\alpha) \approx \frac{12\pi R}{v}\cdot\frac{\gamma^2}{2}\cdot w_\alpha \approx \frac{12\pi R}{v}\cdot\frac{\sinh^2\gamma}{2}\cdot w_\alpha = \frac{6\pi R}{c^2}\cdot\frac{v^{*2}}{R} =$$
$$= \frac{6\pi R}{c^2}\cdot v^*\cdot w_\alpha^* = \frac{6\pi R}{c^2}\cdot \overset{\perp}{g} = \frac{6\pi R}{c^2}\cdot\frac{fM}{R^2} = \boxed{6\pi\cdot\frac{fM}{R\cdot c^2} = 3\pi\cdot\frac{2fM}{R\cdot c^2}} = 3\pi\cdot\frac{\Delta\Pi_{\sigma(f)}}{c^2} + 3\pi\cdot\frac{\Delta\Pi_{\sigma(a)}}{c^2} = 6\pi\cdot\frac{\Delta\Pi_\sigma}{c^2} > 0. \quad (55)$$

(Here v* is the 1st cosmic velocity of the Mercury from the Sun's gravity.)

Gerber's formula, repeated by Einstein traditionally without reference, turned out to be relativistic in P^{3+1} as purely trigonometric! The latter phenomenon is explained here simply by the fact that it is homogeneous and isotropic in nature. Tensor, vector, and scalar pseudo-Euclidean trigonometry are its natural tools! The eccentricity of the orbit in formula (54) is not needed! But it is important for observation and for estimating the average radius as $R=a\cdot(1-e^2)$.

In relativistic physics, it is generally accepted that Newton's classical Principle of Equivalence of inertial and gravitational masses, or acceleration and gravity tension, applies in both non-relativistic and relativistic forms. It has been tested by a number of experimenters, starting with Newton [23]. However, no one has experimentally established: "Whether and how this great Principle applies in a system with a stationary gravitating supermass M and a mass m moving in orbit?”. Above, we found that the concept of inertial mass m_0 in the normal direction to its motion and in the same normal direction of tension g_f (different from the parallel mass m of Mercury in orbit) is applicable as to the gravitational mass m_0. Then Newton's Principle of Equivalence for normal and parallel inertial and gravitational masses m_0 is not violated here. Moreover, nor is Herglotz's Principle from TR kinematics violated, and it also applies in its dynamics and in the theory of gravity, but only in P^{3+1}!

The most interesting thing about GR-effects is that, apart from the Mercury perihelion shift, the rest of these effects have nothing to do with relativity, except for the constancy of the light speed, which was the same and before, according to the super-accurate experiments of Michelson and Morley, and theoretically, according to the hyperbolic angle of motion of Poincaré [2]. They fit into Newton's still immortal theories [26] with the Planck part of QM in P^{3+1}.

In Minkowski space-time P^{3+1}, light rays propagate specifically along the hypersurface of a 3-dimensional isotropic cone at an infinite angle of motion, represented better in the universal base E_1 (for pseudo- and quasi-Euclidean geometries) with an inclination φ_R $(\gamma)=\pi/4$ to the reference axis {ct} and to $E^{3(1)}$. Light lines on an isotropic cosine do not have hyperbolic curvature, since the tangential addition to the energy of photons only leads to an increase in their frequency ν and a decrease in $\lambda=c/\nu$.

However, light lines can have normal orthospherical curvature $K_\nu=d\alpha/R_K$ according to normal inner accelerations of photons, which are Euclidean (see above), and it does not contradict TR in P^{3+1}. Since light propagates as the free motion of photons, then the normal intensity g_f in (50) corresponds to the equivalent normal inner acceleration $g_a=g_\nu$ in (49), which leads to a double normal bending of the light beam from the star. This effect is revealed when a beam of light from a star passes near the Sun's disk during a total solar eclipse.

To estimate the effect in P^{3+1} we calculate [10, p. 261], [9, p. 298], first, using Soldner's method [25] (1804) and Newton's Laws [23], the gravitational deflection of the light ray from its straight trajectory with two equal changes in the gravitational potential of the Sun from zero to $\Delta\Pi_f\{max\}$ and back to zero, which gives normal spherical shifts of the ray towards the Sun's barycenter M:

$$d\delta_I = dl/\overset{\perp}{R} \approx d(-r\cdot\cos\varepsilon)/\overset{\perp}{R} = b\, d(-\cot\varepsilon)/\overset{\perp}{R} = [fM/(bc^2)]\cdot\sin\varepsilon\, d\varepsilon = \Delta\Pi_f(\varepsilon)/c^2\, d\varepsilon = d[2fM/(b\cdot c^2)],$$

$$\delta = 2k_E = 2\cdot\Delta\Pi_f max/c^2 \text{ rad} \approx [fM/(b\cdot c^2)]\cdot\int_0^\pi \sin\varepsilon\, d\varepsilon = 2fM/(b\cdot c^2).$$

(b = const is distance between barycenter M and intersection point of this light ray two asymptotes.)

Secondly, in addition to the Newtonian part, we also calculate the refractive part of the gravitational deflection of the light ray as if by Snellius optical Law, but only as an analogy to the deflection of light due to a change in the refractive index of light, which was previously discussed in Möller's test-book [26, p. 308]. We have this deflection of the ray at the constant speed of light "c" and with the change in the frequency of photons according to the Planck-Einstein formula $\Delta E = h\cdot\Delta\nu = hc/\Delta\lambda$. The frequency of photons ν increases in the first part of their trajectory and decreases in the second part with a ratio for $\Delta\nu$ and with reverse changes in $\Delta\lambda$. The angle of incidence is $+\varepsilon$, if $\varepsilon<\pi/2$; and the angle of incidence is $(\pi-\varepsilon)$ if $\varepsilon>\pi/2$. This interprets the normal shifts of the light ray, in addition to the equal to it Soldner's bending also in the direction of the barycenter M:

$$\sin\varepsilon/\sin(\varepsilon - d\delta_{II}) = \tfrac{\nu+d\nu}{\nu},\ \varepsilon\le\pi/2;\ \sin(\pi-\varepsilon)/\sin(\pi-\varepsilon+d\delta_{II}) = \tfrac{\nu-d\nu}{\nu},\ \varepsilon>\pi/2\ \rightarrow$$

$$\rightarrow d\delta_{II} = \pm d\nu/\nu = d\overset{\perp}{c}/c = \overset{\perp}{g}\, d\tau/c = \overset{\perp}{g}\, dl/c^2 = dl/\overset{\perp}{R} = d\delta_I.$$

Consequently, we have substantiated above both the phenomenon of light refraction in a gravitational field and the Snellius sine Law for it, but *with its peculiarity* above for the gravitational evaluation in P^{3+1}! Then, for this effect, consisting of both Newtonian and refractive equal parts, but assuming a constant speed of light in P^{3+1}, we have the next summary result:

$$2\delta = 4k_E = 2\cdot\Delta\Pi_f max/c^2 + 2\cdot\Delta\Pi_a min/c^2 = 4\cdot\Delta\Pi_f max/c^2 \text{ rad.} \qquad (56)$$

The refractive part is caused by the constant accelerational potential $\Delta\Pi_a = \Delta\Pi_f$.

It is believed that the "red shift" effect for light coming to Earth from the Sun was predicted by Einstein as GR-effect in 1916. However, it was first predicted by John Mitchell (Jean Michel) in his letter to "The London Royal Society" in 1783 [27]. In fact, like the previous one, it is *a non-relativistic effect* of Newton Laws and Planck KM, without relativistic time dilation, since the gravitational cosine in it is equal to 1 due to the coincidence of the direction of photon motion and the gravity intensity. Therefore, this effect *is* actually *single*! Therefore, as a non-relativistic effect, it is correctly described by the action of the Sun gravity intensity on photons with a primary change in their frequency and wave length according to the Planck–Einstein quantum formula for photons depending on the gravitational potential of the Sun and at constant light speed c=ν·λ in vacuum:

$$E_L = h\nu = m_L c^2 = \dot{E}_L - \Delta\Pi_f \cdot m_L = h\dot{\nu} - \Delta\Pi_f \cdot m_L < h\dot{\nu} \Rightarrow \nu < \dot{\nu},\ \ \lambda > \dot{\lambda}. \qquad (57)$$

This now well-known idea was first proposed by Max Born in [28] – one of the founders of the Quantum mechanics. It also has our additional trigonometric condition $\cosh\gamma_{(f)}=1$, i.e., that this effect is single and non-relativistic in P^{3+1}!

Real space-time ("thing in itself" according to Kant), distorted for an external Observer by gravity as *observable*, is represented in *bimetric theories up to the second order of approximation by space-time metric*. This is distorting global gravitational lensing of light. According to Zwicky's idea, this is the loss of energy hν=hc/λ by photons due to the fact that they fly not in empty space. This is the increasing difference in time moments for different galaxies. Distorting factors should act as follows: the farther away cosmic objects are, the greater their contribution will be. The Hubble's Law indirectly confirms all this.

The Hubble's Law in its original celestial form as Δλ/λ=–hΔν/ν=Hl/c=Ht, with the author's interpretation, applies *precisely* to the connection between the redshift of galaxies and the distances to them, initially through parallaxes – *without any his interpretations*, but with next various fantasies on this them.

As for GTR, when applied globally in its space, the parallax method based on the classical trigonometry cannot be used to measure astronomical distances due to the presumed curvature not only of the time-arrow but and of the Euclidean 3D subspace. GTR also has a number of other irreconcilable contradictions.

This is a denial of the Law of energy conservation as an exact Law of Nature. David Hilbert [29] was the first to write about this, and it was then rigorously confirmed theoretically by Noether's Theorem, his post-student in Göttingen [1]. It is the fact that in GTR there are no *multiple* transformations of space-time coordinates that guarantee correctly obtaining of unambiguous interpretations independent on them – similar to Lorentz invariance in Minkowski space-time, and there is not even a real origin of coordinates that constantly shifts with *the change* in the structure of space. It is that its local pseudo-Riemannian curvature tensor causes the curvature not only of the time coordinate, but also of all spatial coordinates with stationary material objects in the Universe! Where, then, are the incredible mechanical stresses in these objects? The geometric parameters of stationary objects are invariant! This is an inviolable axiom, sacredly observed by in P^{3+1}. This is also the theoretical possibility of realizing closed time-like trajectories in GTR, which violates the sacred Principle of Determinism; in other words, it promises GTR apologists meeting with their ancestors or descendants! In GTR interpretation, pseudo-Riemannian space-time is constantly expanding with acceleration, and the further away, the faster *without limit on the speed* "c"! Relativistic Quantum Mechanics and Higgs' Theory of inertia of matter are presented in P^{3+1}! The Higgs' theory, the generation of matter was accompanied by the appearance of the global field of inertia. Consequently, and Ernst Mach was right in qualitatively explaining the inertia of mass by the influence of the matter of the Universe – Mach's Principle [30]), which GTR ultimately rejected.

However, we hope that this article will contribute to resolving the acute problem of Theory of Relativity compatibility with other fundamental theories!

Inferences

1. The problem of incompatibility between GRT and Relativistic QM is leveled with the application of the Tensor Trigonometry: it is sufficient for TR to be compatible with gravity and RQM in the Minkowski space-time P^{3+1}.
2. In P^{3+1} the general 4x4-tensors of energy-momentum, momenta, velocity, and potential are introduced, which are proportional to the general 4x4-tensor of rotations and identical to the tensor of motions in the Lobachevsky geometry.

3. In P^{3+1} the new formulas and theorems for summing motions and physical relativistic velocities are given in a complete form with polar decomposition.
4. P^{3+1} all new formulas and theorems for summing differential motions, internal accelerations, and rotation with the Thomas precession are given.
5. The application of physical formulas of tensor trigonometry for travel in the solar system and deep cosmos is considered, their reality is discussed, and the Ziolkovsky formula is given in relativistic trigonometric form.
6. The precise trigonometric formulas are given for the main GR-effects.
7. The own and total energy of matter E_0 and E – inertial, electromagnetic, and gravitational are generated by actions of the reference gravitational potential of the Universe as $\Pi_0=c^2$ on the mass m_0 and m.
8. The speed of light is not strictly as the absolute constant in theory, but is related to the reference gravitational potential of the Universe Π_0, which depends on the distribution of matter in the Universe relative to a world point.
9. The speed of light could have changed significantly during critical phases of change in the material world, and it can increase noticeably in the vicinity of extremely massive and very dense cosmic objects, including near and inside the so-called "black holes."
10. The Theory of Relativity in P^{3+1} with the Poincaré group approach is an exact science, isomorphic to the pseudo-Euclidean Tensor Trigonometry at $v< c$ and under $c=const$.

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25 January 2026

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