Euler brick.

S.G.Bratzev

In [1] about this task, reads as follows: "Among the many geometrical Diophantine problems whose solution has been found so far, one of the most difficult and the most famous is that of the so-called" brick integer "or" Euler brick ". Under the "brick" in this case refers to a cuboid. At the same time we have seven unknowns: three ribs brick, three of his facial diagonal and one space diagonal passing through the center of the brick from one of its corner to another (Figure 1.).

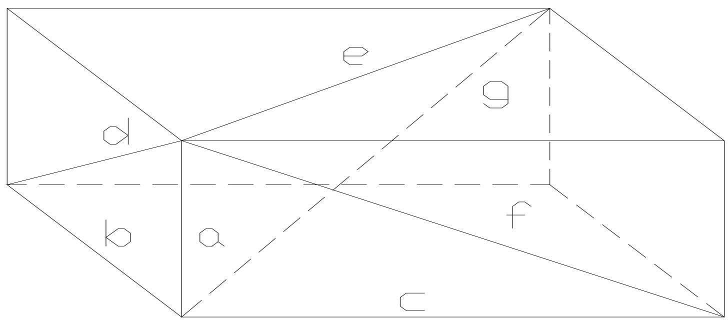

Fig. 1. Brick integers - Diophantine problems unsolved.

Whether there will be a brick, in which all seven of these variables are integers?

This problem is equivalent to finding the integer solutions of the following system of equations with seven unknowns:

$$\begin{cases} a^2 + b^2 = d^2, \\ b^2 + c^2 = e^2, \\ a^2 + c^2 = f^2, \\ a^2 + e^2 = g^2. \end{cases} \qquad (1)$$

This problem has not been solved so far, as not yet found evidence that its solution does not exist. "

This problem is also discussed in the book [1]. Overview of the problem is presented in [3]. It is not known whether this task considering Diophantus, but Euler already knew her. Thus, this problem is not less than 250 years.

Let us assume that $a \le b \le c$.

Consider the first three equations of (1):

$$\begin{cases} a^2 + b^2 = d^2, \\ b^2 + c^2 = e^2, \\ a^2 + c^2 = f^2. \end{cases} \qquad (2)$$

This system of equations defines three interrelated Pythagorean triangle. A plan for solving the problem is as follows: find a set of solutions of (2), and then select a subset of it, which satisfies the system (1); If this set is empty - solution of the diophantine "brick" does not exist.

Instead of (1) to (2) we add the equation

$$a^2 + b^2 + c^2 = g^2. \qquad (3)$$

Solutions of each of the equations (2) consistently have the following form:

$$\begin{cases} a = 2p_1q_1, & b = p_1^2 - q_1^2, & d = p_1^2 + q_1^2, \\ b = p_2^2 - q_2^2, & c = 2p_2q_2, & e = p_2^2 + q_2^2, \\ a = 2p_3q_3, & c = p_3^2 - q_3^2, & f = p_3^2 + q_3^2, \end{cases} \qquad (4)$$

Where

$$\begin{cases} p_1 = x_1\sqrt{\alpha_1}, & q_1 = y_1\sqrt{\alpha_1}, \\ p_2 = x_2\sqrt{\alpha_2}, & q_2 = y_2\sqrt{\alpha_2}, \\ p_3 = x_3\sqrt{\alpha_3}, & q_3 = y_3\sqrt{\alpha_3}, \end{cases} \qquad (5)$$

where x_1, x_2, x_3, y_1, y_2, y_3 - integers;

α_1. α_2. α_3 - integers that are not perfect squares (except when any of them is equal to 1), and all α_i, $1 \le i \le 3$ are relatively prime to each other.

(4) with (5) we obtain the system:

$$\begin{cases} \alpha_1 x_1 y_1 = \alpha_3 x_3 y_3, \\ \alpha_1 (x_1^2 - y_1^2) = \alpha_2 (x_2^2 - y_2^2), \\ 2\alpha_2 x_2 y_2 = \alpha_3 (x_3^2 - y_3^2). \end{cases} \qquad (6)$$

Solving this system with respect to y1, y2, y3, we get:

$$\begin{cases} y_1 = \frac{1}{\alpha_1 x_1^2} \sqrt{G - 2\alpha_3 x_2 x_3^2 \sqrt{R}}, \\ y_2 = \frac{1}{\alpha_1 \alpha_2 x_1^2} (-\alpha_2 \alpha_3 x_2 x_3^2 + \sqrt{R}), \\ y_3 = \frac{1}{\alpha_3 x_1 x_3} \sqrt{G - 2\alpha_3 x_2 x_3^2 \sqrt{R}}, \end{cases} \qquad (7)$$

Where

$$\begin{cases} G = \alpha_3^2 x_3^4 (2\alpha_2 x_2^2 + \alpha_1 x_1^2) + (\alpha_1 - 1)\alpha_1 x_1^4 (\alpha_2 x_2^2 - \alpha_1 x_1^2), \\ R = \alpha_2 \left[\alpha_2 x_2^2 (\alpha_3^2 x_3^4 + \alpha_1^2 x_1^4) + \alpha_1 x_1^2 (\alpha_3^2 x_3^4 - \alpha_1^2 x_1^4) \right]. \end{cases} \quad (8)$$

From the expression for R (8) it follows that in order to R had a perfect square must exist integers λ_1 and λ_2 such that the following equation:

$$\begin{cases} \alpha_3^2 x_3^4 - \alpha_1^2 x_1^4 = \alpha_2 \lambda_1, \\ x_2^2 (\alpha_3^2 x_3^4 + \alpha_1^2 x_1^4) + \alpha_1 \lambda_1 x_1^2 = \lambda_2^2 \end{cases} \quad (9)$$

The second of these equations can be represented as:

$$\left[x_2 (\alpha_3 x_3^2 - \alpha_1 x_1^2) \right]^2 + (2\alpha_1 \alpha_3 x_2^2 x_3^2 + \lambda_1) x_1^2 = \lambda_2^2. \quad (10)$$

The last equation can be rewritten as:

$X^2 + AY^2 = Z^2$

(eleven)

Where

$$\begin{cases} X = x_2(\alpha_3 x_3^2 - \alpha_1 x_1^2), \\ Y = x_1, \\ Z = \lambda_2, \\ A = 2\alpha_1\alpha_3 x_2^2 x_3^2 + \lambda_1. \end{cases} \tag{12}$$

Solution of (11.11) are of the formula:

$$\begin{cases} X = n^2 - Am^2, \\ Y = 2nm, \\ Z = n^2 + Am^2, \end{cases} \tag{13}$$

where n, m - and integers (n, m) = 1.

Hence it is easy to obtain the following formula for λ_1 and λ_2:

$$\begin{cases} \lambda_1 = \dfrac{n^2}{\alpha_1 m^2} - 2\alpha_3 x_2^2 x_3^2 - \dfrac{2n\alpha_3 x_2 x_3^2}{m\alpha_1 x_1} + \dfrac{2n}{m} x_1 x_2, \\ \lambda_2 = \dfrac{n}{m} x_1 - \alpha_3 x_2 x_3^2 + \alpha_1 x_1^2 x_2. \end{cases} \tag{14}$$

From the first formula in (14) it follows thatα_1 = 1 and m = 1. Then formulas (14) take the form:

$$\begin{cases} \lambda_1 = n^2 - 2\alpha_3 x_2^2 x_3^2 - \dfrac{2n\alpha_3 x_2 x_3^2}{x_1} + 2nx_1 x_2, \\ \lambda_2 = nx_1 - \alpha_3 x_2 x_3^2 + x_1^2 x_2. \end{cases} \quad (15)$$

Furthermore, (7) it follows that in order to expression $G - 2\alpha_3 x_2 x_3^2 \sqrt{R}$ it was a perfect square, it is necessary the existence of a λ_3Which satisfies:

$$\alpha_3 x_3^2 (2\alpha_2 x_2^2 + x_1^2) - 2\alpha_2 x_2 \lambda_2 = \alpha_3 \lambda_3^2. \quad (16)$$

Hence:

$$\lambda_3^2 = 4\alpha_2 x_2^2 x_3^2 + x_1^2 x_3^2 - \frac{2\alpha_2}{\alpha_3} x_1^2 x_2^2 - \frac{2n\alpha_2}{\alpha_3} x_1 x_2$$

(17)

But since (α_3, X1) = (α_3X2) = 1 then from (17) it follows thatα_3 = 2. Consequently,

$$\lambda_3^2 = 4\alpha_2 x_2^2 x_3^2 + x_1^2 x_3^2 - \alpha_2 x_1^2 x_2^2 - n\alpha_2 x_1 x_2 . \qquad (18)$$

Using (15) and (16), we obtain:

$$\begin{cases} \sqrt{R} = \alpha_2 \lambda_2 , \\ \sqrt{G - 4x_2 x_3^2 \sqrt{R}} = 2x_3 \lambda_3 \end{cases} \qquad (19)$$

Substituting in these formulas (7), we obtain the following representation for y_1, y_2, y_3:

$$\begin{cases} y_1 = \dfrac{2x_3 \lambda_3}{x_1^2}, & (20) \\ y_2 = x_2 + \dfrac{nx_1 - 4x_2 x_3^2}{x_1^2}, & (21) \\ y_3 = \dfrac{\lambda_3}{x_1} & (22) \end{cases}$$

From formula (21) shows that x_1 and x_3 = n =ωx1 whereω- an integer. Let x = $x_1 = x_3$. Thenλ_1. λ_2. λ_3 we obtain the following expression:

$$\begin{cases} \lambda_1 = x^2(\omega^2 - 2\omega x_2 - 4x_2^2), \\ \lambda_2 = x^2(\omega - x_2), \\ \lambda_3^2 = x^2(x^2 + 3\alpha_2 x_2^2 - \omega\alpha_2 x_2). \end{cases} \quad (23)$$

designating $h^2 = x^2 + 3\alpha_2 x_2^2 - \omega\alpha_2 x_2$, We obtain:

$$y_1 = 2h, \qquad y_2 = \frac{x^2 - h^2}{\alpha_2 x_2}, \qquad y_3 = h. \quad (24)$$

Then, formula (4) take the form:

$$\begin{cases} a = 4xh, \quad b = x^2 - 4h^2, \quad d = x^2 + 4h^2, \\ b = r - \frac{(x^2 - h^2)^2}{r}, \quad c = 2(x^2 - h^2), \quad e = r + \frac{(x^2 - h^2)^2}{r}, \\ f = 2(x^2 + h^2), \end{cases} \quad (25)$$

Where $r = \alpha_2 x_2^2$.

Now it is easy to prove that the formula (25) describes a number of solutions of the system (2) if and only if the following equation:

$$\begin{cases} r^2-(x^2-4h^2)r-(x^2-h^2)^2=0, & (26) \\ (x^2-h^2)^2=\tau r, & (27) \end{cases}$$

Where τ- an integer. Solving equation (26) with respect to r, we obtain:

$$r=\frac{1}{2}\left[(x^2-4h^2)+\sqrt{(x^2-4h^2)^2+4(x^2-h^2)^2}\right] \tag{28}$$

It follows that there must be an integer C, for which the equality

$$(x^2-4h^2)^2+\left[2(x^2-h^2)\right]^2=C^2 \tag{29}$$

The solutions of (29) have the form:

$$\begin{cases} x^2 - 4h^2 = \alpha(k^2 - \ell^2), \\ x^2 - h^2 = \alpha k\ell, \\ C = \alpha(k^2 + \ell^2) \end{cases} \qquad \text{(thirty)}$$

Where α, k, ℓ - whole numbers.

From this and (28) that $\alpha k^2 = \alpha_2 x_2^2$, i.e $\alpha = \alpha_2$, K = x2. Finally, from the first system (30) of two equations that

$$h = \sqrt{\frac{\alpha}{3}(k\ell - k^2 + \ell^2)} \,. \qquad (31)$$

$$x = \sqrt{\frac{\alpha}{3}(4k\ell - k^2 + \ell^2)} \qquad (32)$$

here $\alpha = \alpha_2 = 3$, r = 3k2

Substituting (31), (32) and the last representation for r in (25), we find that the solution (2) is described by the following formulas:

$$\begin{cases} a = 4\sqrt{(k\ell - k^2 + \ell^2)(4k\ell - k^2 + \ell^2}, \\ b = 3(k^2 - \ell^2), \\ c = 6k\ell, \\ d = 8k\ell - 5(k^2 - \ell^2), \\ e = 3(k^2 + \ell^2), \\ f = 2\left[5k\ell - 2(k^2 - \ell^2)\right] \end{cases} \qquad (33)$$

where the integers k and ℓ such that the value

$$\begin{cases} h^2 = k\ell - k^2 + \ell^2 \\ x^2 = 4k\ell - k^2 + \ell^2 \end{cases} \qquad (34)$$

a perfect square

It is not difficult to prove that the system (34) is equivalent to the system (35)

$$\begin{cases} (x-h)(x+h) = 3kl \\ (x-2h)(x+2h) = 3(k+l)(k-l) \end{cases} \qquad (35)$$

Further, the first equation (35) can be replaced by one of the following system of equations:

$$\begin{cases} x - h = 3\alpha k \\ x + h = \frac{l}{\alpha} \end{cases}; \quad \begin{cases} x - h = \alpha k \\ x + h = \frac{3l}{\alpha} \end{cases}; \quad \begin{cases} x - h = 3\alpha l \\ x + h = \frac{k}{\alpha} \end{cases};$$

$$\begin{cases} x - h = \alpha l \\ x + h = \frac{3k}{\alpha} \end{cases}; \qquad (36)$$

A second equation system (35) can be replaced by one of the following systems:

$$\begin{cases} x-2h=3\beta(k-l) \\ x+2h=\frac{k+l}{\beta} \end{cases};\quad \begin{cases} x-2h=\beta(k-l) \\ x+2h=\frac{3(k+l)}{\beta} \end{cases};\quad \begin{cases} x-2h=3\beta(k+l) \\ x+2h=\frac{k-l}{\beta} \end{cases};$$

$$\begin{cases} x-2h=\beta(k+l) \\ x+2h=\frac{3(k-l)}{\beta} \end{cases} \qquad (37)$$

Where α and β - unknown rational numbers.

Combining systems (36) and (37) gives 16 variants.

1 embodiment.

$$\begin{cases} x-h=3\alpha k \\ x+h=\frac{l}{\alpha} \\ x-2h=3\beta(k-l) \\ x+2h=\frac{k+l}{\beta} \end{cases} \qquad (38)$$

Investigation of the system (38) carry out on the manner in which we explore the rest of the 15 variants. Therefore, this scheme we describe here in detail.

Sequentially stacking the first two equations (38); Subtracting the first equation from the second; Folding the third and fourth equations and subtracting the third equation of the fourth, we get:

$$\begin{cases} 2x=3\alpha(k-l)+\frac{k+l}{\beta} \\ 4h=-6\alpha k+\frac{2l}{\alpha}=-3\beta(k-l)+\frac{k+l}{\beta} \end{cases} \qquad (39)$$

Further, the folding system of equations (39) and subtracting the second equation from the first we obtain

$$\begin{cases} -3\alpha k+\frac{3l}{\alpha}=\frac{2(k+l)}{\beta} \\ 9\alpha k-\frac{l}{k}=6\beta(\gamma-1) \end{cases} \qquad (40)$$

Denoting γ =, the system (40) in the form$\frac{k}{l}$

$$\begin{cases} -3\alpha\gamma+\frac{3}{\alpha}=\frac{2(\gamma+1)}{\beta} \\ 9\alpha\gamma-\frac{1}{\alpha}=6\beta(\gamma-1) \end{cases} \qquad (41)$$

Or

$$\begin{cases} -3\alpha^2\beta\gamma + 3\beta = 2\alpha\gamma + 2\alpha \\ 9\alpha^2\gamma - 1 = 6\alpha\beta\gamma - 6\alpha\beta \end{cases} \quad (42)$$

here

$$\gamma = \frac{3\beta - 2\alpha}{\alpha(2+3\alpha\beta)} = \frac{6\alpha\beta - 1}{3\alpha(2\beta - 3\alpha)} \quad (43)$$

Given that from (35) k> l, we get γ> 1.

Then from (43)

$$\begin{cases} \beta > \frac{4\alpha}{3(1-\alpha^2)} \\ 9\alpha^2 > 1 \end{cases} \quad (44)$$

Here.$\frac{1}{3} < \alpha < 1$

Next, from (43) it follows that

$$\frac{3\beta - 2\alpha}{2+3\alpha\beta} = \frac{6\alpha\beta - 1}{3(2\beta - 3\alpha)}$$

here

18α2 + 18β2-18α2β2 + 2-48αβ = 0,

Or

9α2 + 9β2-9α2β2 + 1-24αβ = 0.

This is equivalent to

(3α-3β) 2 = 9α2β2 + 6αβ-1,

Or

(3α-3β) 2 = (3αβ + 1) 2-2.

here

(3αβ + 1) 2- (3α-3β) 2 = 2,

(45)

which is equivalent to

(3αβ + 1-3α + 3β) (3αβ + 1 + 3α-3β) = 2

(46)

Equation (46) is split into two systems:

$$\begin{cases} 3\alpha\beta + 1 - 3\alpha + 3\beta = 1 \\ 3\alpha\beta + 1 + 3\alpha - 3\beta = 2 \end{cases} \quad (47)$$

Or

$$\begin{cases} 3\alpha\beta + 1 - 3\alpha + 3\beta = 2 \\ 3\alpha\beta + 1 + 3\alpha - 3\beta = 1 \end{cases} \quad (48)$$

The system (47) has a solution of α = 1/2, β = 1/3 but in this case, γ = 0, ie. k = l, which is impossible. The system as (48) has a solution of α = 1/3, β = 1/2, in this case k = 1, which is also impossible. Therefore, the original system (38) has no solution and the embodiment 1 is impossible.

Option 2.

$$\begin{cases} x - h = 3\alpha k \\ x + h = \frac{l}{\alpha} \\ x - 2h = \beta(k - l) \\ x + 2h = \frac{3(k+l)}{\beta} \end{cases} \quad (49)$$

Now we write the analog of formula (43):

$$\gamma = \frac{\beta - 2\alpha}{\alpha(\alpha\beta + 2)} = \frac{2\alpha\beta - 1}{\alpha(2\beta - 9\alpha)} \quad (50)$$

From this and γ> 1 should be

$$\begin{cases} \beta > \frac{4\alpha}{1-\alpha^2} \\ 9\alpha^2 > 1 \end{cases}$$

Those. 1/3 <α <1

Analogue of (45) has the form

(Aβ + 1) 2- (3α-β) 2 = 2

Those.

$$\begin{cases} \alpha\beta + 1 - 3\alpha + \beta = 1 \\ \alpha\beta + 1 + 3\alpha - \beta = 2 \end{cases} \quad (51)$$

Or

$$\begin{cases} \alpha\beta + 1 - 3\alpha + \beta = 2 \\ \alpha\beta + 1 + 3\alpha - \beta = 1 \end{cases} \quad (52)$$

The solution of (51): α = 1/2, β = 1, i.e. γ = 0, which is impossible.

The solution of (52): α = 1/3, β = 3/2, i.e. γ = 1, which is also impossible.Hence embodiment 2 is eliminated.

Option 3.

$$\begin{cases} x - h = 3\alpha k \\ x + h = \frac{l}{\alpha} \\ x - 2h = 3\beta(k + l) \\ x + 2h = \frac{k-l}{\beta} \end{cases} \quad (53)$$

According to the proposed scheme

$$\gamma = \frac{2\alpha+3\beta}{\alpha(2+3\alpha\beta)} = \frac{1+6\alpha\beta}{3\alpha(3\alpha-2\beta)} \quad (54)$$

From this and $\gamma > 1$ implies $1/3 < \alpha < 1$, $0 < \beta < 2/3$. (55)

Further, from the equation

$$\frac{2\alpha+3\beta}{2+3\alpha\beta} = \frac{1+6\alpha\beta}{3(3\alpha-2\beta)}$$

must be

$$\beta^2 = \frac{9\alpha^2-1}{9(\alpha^2+1)}.$$

Or

$$(3\alpha - 1)(3\alpha + 1) = 9\beta^2(\alpha^2 + 1) \quad (56)$$

This equation is split into eight options:

$$\begin{cases} 3\alpha - 1 = 9\beta^2 \\ 3\alpha + 1 = 1 + \alpha^2 \end{cases} \quad (57)$$

Or

$$\begin{cases} 3\alpha - 1 = 9(1 + \alpha^2) \\ 3\alpha + 1 = \beta^2 \end{cases} \quad (58)$$

Or

$$\begin{cases} 3\alpha - 1 = 1 + \alpha^2 \\ 3\alpha + 1 = 9\beta^2 \end{cases} \quad (59)$$

Or

$$\begin{cases} 3\alpha - 1 = \beta^2 \\ 3\alpha + 1 = 9(1 + \alpha^2) \end{cases} \quad (60)$$

Or

$$\begin{cases} 3\alpha - 1 = 9\beta \\ 3\alpha + 1 = \beta(1 + \alpha^2) \end{cases} \quad (61)$$

Or

$$\begin{cases} 3\alpha - 1 = 9\beta(1 + \alpha^2) \\ 3\alpha + 1 = \beta \end{cases} \quad (62)$$

Or

$$\begin{cases} 3\alpha - 1 = \beta(1 + \alpha^2) \\ 3\alpha + 1 = 9\beta \end{cases} \quad (63)$$

Or

$$\begin{cases} 3\alpha - 1 = \beta \\ 3\alpha + 1 = 9\beta(1 + \alpha^2) \end{cases} \quad (64)$$

Computing systems solutions sequentially (57) ÷ (70), we obtain:

1. For the system (57) α = 3, which is contrary to condition (55).
2. For the system (58) $\alpha_{1,2}$ = 1/18 (3 ±), i.e. real solutions do not exist.$\sqrt{-351}$
3. For the system (59) $\alpha_1 = 1$, $\alpha_2 = 2$, which again contradicts (55);
4. For the system (60), i.e. no real solutions;$\alpha_{1,2} = \frac{1}{18}(3 \pm \sqrt{-281})$
5. cubic equation is obtained for the system (61)

$$3\alpha^3 - \alpha^2 - 24\alpha - 10 = 0 \quad (65)$$

The above equation after substitution has the form$\alpha = y + \frac{1}{9}$

$$y^3 - \frac{217}{27}y - \frac{344}{81} = 0 \quad (66)$$

The discriminant of this equation

$$Q = -\frac{1}{27} \cdot \left(\frac{217}{27}\right)^3 + \frac{1}{4} \cdot \left(\frac{344}{81}\right)^2 = \frac{2349215}{531441} > 0$$

Hence equation (66) has one real root. Not hard to figure out that this is the root

$$y \approx \sqrt[3]{4.123} + \sqrt[3]{0.123}$$

Hence root of the equation (65)

$$\alpha \approx 0{,}111 + \sqrt[3]{4{,}123} + \sqrt[3]{0{,}123}$$

It is obvious that α> 1, which contradicts the condition (55).

6. For the system (62), we obtain a cubic equation

$$27\alpha^3 + 9\alpha^2 + 24\alpha + 10 = 0$$

Clearly, this equation has no positive real roots.

7. For the system (63), we obtain a cubic equation

$3\alpha^3 + \alpha^2 + 24\alpha - 10 = 0$

This equation also has no positive real roots.

8. For the system (64) we obtain a cubic equation

$$27\alpha^3 - 9\alpha^2 + 24\alpha - 10 = 0 \tag{67}$$

The above equation after substitution has the form$\alpha = y + \frac{1}{9}$

$$y^3 - \frac{1}{27}y - \frac{200}{729} = 0 \tag{68}$$

The discriminant of this equation

$Q = \frac{9271}{531441} \approx 0.017 > 0$

Hence equation (68) has one real root

$y \approx -\left(\sqrt[3]{0.005} + \sqrt[3]{0.269}\right)$

Hence the root of the equation (67)

$\alpha \approx 0{,}111 - \sqrt[3]{0{,}005} - \sqrt[3]{0{,}269}$

It is clear that, contrary to the condition (55).$\alpha < \frac{1}{3}$

Therefore, the embodiment 3 is not possible.

Option 4.

$$\begin{cases} x - h = 3\alpha k \\ x + h = \frac{l}{\alpha} \\ x - 2h = \beta(k + l) \\ x + 2h = \frac{3(k-l)}{\beta} \end{cases} \tag{69}$$

compute

$$\gamma = \frac{2\alpha+\beta}{\alpha(2+\alpha\beta)} = \frac{1+2\alpha\beta}{\alpha(9\alpha-2\beta)} \tag{70}$$

From this and γ> 1 we obtain

$$\alpha < 1 \tag{71}$$

from equation

$\frac{2\alpha+\beta}{2+\alpha\beta} = \frac{1+2\alpha\beta}{9\alpha-2\beta}$

must be

$$(3\alpha - 1)(3\alpha + 1) = \beta^2(1 + \alpha^2) \tag{72}$$

There are four possibilities:

$$\begin{cases} 3\alpha - 1 = \beta^2 \\ 3\alpha + 1 = 1 + \alpha^2 \end{cases} \quad (73)$$

Or

$$\begin{cases} 3\alpha - 1 = 1 + \alpha^2 \\ 3\alpha + 1 = \beta^2 \end{cases} \quad (74)$$

Or

$$\begin{cases} 3\alpha - 1 = \beta \\ 3\alpha + 1 = \beta(1 + \alpha^2) \end{cases} \quad (75)$$

Or

$$\begin{cases} 3\alpha - 1 = \beta(1 + \alpha^2) \\ 3\alpha + 1 = \beta \end{cases} \quad (76)$$

1. For the system (73) solution of α = 3, which is contrary to condition (71);
2. For the system (74) the solution α1 = 1, α2 = 2, which is also contrary to condition (71);
3. For the system (75), we obtain a cubic equation

$$3\alpha^3 - \alpha^2 - 2 = 0 \quad (77)$$

It is easy to prove that the discriminant of this equation Q> 0. Hence equation (77) has one real root. It is clear that the root α = 1, and this is contrary to condition (71).

4. For the system (76), we obtain a cubic equation

$$3\alpha^3 + \alpha^2 + 2 = 0$$

Clearly, this equation has no real positive roots.

Hence, the embodiment 4 is impossible.

Option 5.

$$\begin{cases} x - h = \alpha k \\ x + h = \frac{3l}{\alpha} \\ x - 2h = 3\beta(k - l) \\ x + 2h = \frac{k+l}{\beta} \end{cases} \quad (78)$$

compute

$$\gamma = \frac{9\beta-2\alpha}{\alpha(2+\alpha\beta)} = \frac{1-2\alpha\beta}{\alpha(\alpha-2\beta)} \quad (79)$$

From this and γ> 1 we obtain 0 <α <1.

from equation

$$\frac{9\beta-2\alpha}{2+\alpha\beta} = \frac{1-2\alpha\beta}{\alpha-2\beta}$$

must be

$$(\alpha\beta+1)^2-(3\beta-\alpha)^2=2$$

Hence, two options:

$$\begin{cases}\alpha\beta+1-3\beta+\alpha=1\\ \alpha\beta+1+3\beta-\alpha=2\end{cases} \quad (80)$$

Or

$$\begin{cases}\alpha\beta+1-3\beta+\alpha=2\\ \alpha\beta+1+3\beta-\alpha=1\end{cases} \quad (81)$$

The system (80) has a solution, contrary to the condition 0 <α <1.$\alpha_1 = -\frac{3}{2}\alpha_2 = 1$

The system (81) has a solution α1 = -1, α2 = 3/2, which is also contrary to the inequalities 0 <α <1.

Thus, the embodiment 5 is not possible.

Option 6.

$$\begin{cases}x-h=\alpha k\\ x+h=\frac{3l}{\alpha}\\ x-2h=\beta(k-l)\\ x+2h=\frac{3(k+l)}{\beta}\end{cases} \quad (82)$$

compute

$$\gamma = \frac{9\beta-6\alpha}{\alpha(6+\alpha\beta)} = \frac{3-2\alpha\beta}{\alpha(3\alpha-2\beta)}$$

The equation

$$\frac{9\beta-6\alpha}{6+\alpha\beta} = \frac{3-2\alpha\beta}{3\alpha-2\beta}$$

equivalent to the equation

$$(\alpha\beta-3)^2-(3\alpha+3\beta)^2=18$$

Or

$$(\alpha\beta-3-3\alpha-3\beta)(\alpha\beta-3+3\alpha+3\beta)=18 \quad (83)$$

This equation is split into the following systems:

$$\begin{cases}\alpha\beta-3-3\alpha-3\beta=1\\ \alpha\beta-3+3\alpha+3\beta=18\end{cases} \quad (84)$$

Or

$$\begin{cases}\alpha\beta-3-3\alpha-3\beta=2\\ \alpha\beta-3+3\alpha+3\beta=9\end{cases} \quad (85)$$

Or

$$\begin{cases}\alpha\beta-3-3\alpha-3\beta=3\\ \alpha\beta-3+3\alpha+3\beta=6\end{cases} \quad (86)$$

1. The system (84) gives the equation

$$6\alpha^2-17\alpha+75=0$$

The roots of this equation.$\alpha_{1,2}=\frac{1}{12}\left(17\pm\sqrt{-1511}\right)$

2. The system (85) gives the equation

   $$6\alpha^2-7\alpha+51=0$$

   roots$\alpha_{1,2}=\frac{1}{12}\left(7\pm\sqrt{-1175}\right)$

3. The system (86) gives the equation

$$2\alpha^2-\alpha+15=0$$

Roots.$\alpha_{1,2}=\frac{1}{4}\left(1\pm\sqrt{-119}\right)$

So, option 6 is not possible.

Option 7.

$$\begin{cases}x-h=\alpha k\\ x+h=\frac{3l}{\alpha}\\ x-2h=3\beta(k+l)\\ x+2h=\frac{k-l}{\beta}\end{cases} \quad (87)$$

We compute

$$\gamma=\frac{2\alpha+9\beta}{\alpha(2+\alpha\beta)}=\frac{1+2\alpha\beta}{\alpha(\alpha-2\beta)}$$

From this and γ> 0, we get

$$\begin{cases} 1 < \alpha < 3 \\ \beta > \frac{\alpha^2 - 1}{4\alpha} \end{cases} \quad (88)$$

The equation

$$\frac{2\alpha+9\beta}{2+\alpha\beta} = \frac{1+2\alpha\beta}{\alpha-2\beta}$$

equivalent to the equation

$$\alpha^2 - 9\beta^2 - \alpha^2\beta^2 = 1$$

Or

$$(\alpha - 1)(\alpha + 1) = \beta^2(9 + \alpha^2)$$

The last equation is split into the system:

$$\begin{cases} \alpha - 1 = \beta^2 \\ \alpha + 1 = 9 + \alpha^2 \end{cases} \quad (89)$$

Or

$$\begin{cases} \alpha - 1 = 9 + \alpha^2 \\ \alpha + 1 = \beta^2 \end{cases} \quad (90)$$

Or

$$\begin{cases} \alpha - 1 = \beta(9 + \alpha^2) \\ \alpha + 1 = \beta \end{cases} \quad (91)$$

Or

$$\begin{cases} \alpha - 1 = \beta \\ \alpha + 1 = \beta(9 + \alpha^2) \end{cases} \quad (92)$$

1. For the system (89) of the equation

$$\alpha^2 - \alpha + 8 = 0$$

It gives a solution - imaginary.$\alpha_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{-31}\right)$

2. For the system (90), the equation

$$\alpha^2 - \alpha + 10 = 0$$

It gives a solution - imaginary.$\alpha_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{-39}\right)$

3. For the system (91), the equation

$9\alpha^3 + \alpha^2 + 8\alpha - 8 = 0$

It has no real positive solutions.

4. For the system (92), the equation

$\alpha^3 - \alpha^2 + 8\alpha - 10 = 0$

This equation has a unique real solution

$\alpha = \frac{1}{3} + \sqrt[3]{7{,}358} - \sqrt[3]{0{,}099}$Which, although satisfies condition (88) but is not rational, since discriminant equation

$Q = \frac{10133}{729}$ and $\sqrt{Q} = \frac{\sqrt{10133}}{27}$

Option 8.

$$\begin{cases} x - h = \alpha k \\ x + h = \frac{3l}{\alpha} \\ x - 2h = \beta(k + l) \\ x + 2h = \frac{3(k-l)}{\beta} \end{cases} \quad (93)$$

compute

$\gamma = \frac{6\alpha+9\beta}{\alpha(6+\alpha\beta)} = \frac{3+2\alpha\beta}{\alpha(3\alpha-2\beta)}$

From the condition $\gamma > 1$ we obtain

$$\begin{cases} 1 < \alpha < 3 \\ \beta > \frac{3(\alpha^2-1)}{4\alpha} \end{cases} \quad (94)$$

Next, the equation

$\frac{6\alpha+9\beta}{6+\alpha\beta} = \frac{3+2\alpha\beta}{3\alpha-2\beta}$

It gives

$9\alpha^2 - 9\beta^2 = 9 + \alpha^2\beta^2$

or

$$\alpha^2(3 - \beta)(3 + \beta) = 9(1 + \beta^2) \quad (95)$$

From (95) we obtain the following 27 options:

$$\begin{cases} \alpha^2(3 - \beta) = 9 \\ 3 + \beta = 11\beta^2 \end{cases} \quad (96)$$

$$\begin{cases}\alpha^2(3-\beta)=11\beta^2\\ 3+\beta=9\end{cases} \quad (97)$$

$$\begin{cases}3-\beta=1+\beta^2\\ \alpha^2(3+\beta)=9\end{cases} \quad (98)$$

$$\begin{cases}3-\beta=9\\ \alpha^2(3+\beta)=1+\beta^2\end{cases} \quad (99)$$

$$\begin{cases}\alpha(3-\beta)=9\\ \alpha(3+\beta)=1+\beta^2\end{cases} \quad (100)$$

$$\begin{cases}\alpha(3-\beta)=1+\beta^2\\ \alpha(3+\beta)=9\end{cases} \quad (101)$$

$$\begin{cases}\alpha^2=9\\ (3-\beta)(3+\beta)=1+\beta^2\end{cases} \quad (102)$$

$$\begin{cases}\alpha^2=1+\beta^2\\ (3-\beta)(3+\beta)=9\end{cases} \quad (103)$$

$$\begin{cases}\alpha=3\\ \alpha(3-\beta)(3+\beta)=3(1+\beta^2)\end{cases} \quad (104)$$

$$\begin{cases}\alpha(3-\beta)(3+\beta)=3\\ \alpha=3(1+\beta^2)\end{cases} \quad (105)$$

$$\begin{cases}\alpha^2(3-\beta)=3\\ 3+\beta=3(1+\beta^2)\end{cases} \quad (106)$$

$$\begin{cases}3-\beta=3\\ \alpha^2(3+\beta)=3(1+\beta^2)\end{cases} \quad (107)$$

$$\begin{cases}\alpha^2(3+\beta)=3\\ 3-\beta=3(1+\beta^2)\end{cases} \quad (108)$$

$$\begin{cases}3+\beta=3\\ \alpha^2(3-\beta)=3(1+\beta^2)\end{cases} \quad (109)$$

$$\begin{cases}\alpha(3-\beta)=3\\ \alpha(3+\beta)=3(1+\beta^2)\end{cases} \quad (110)$$

$$\begin{cases}\alpha(3-\beta)=3(1+\beta^2)\\ \alpha(3+\beta)=3\end{cases} \quad (111)$$

$$\begin{cases}\alpha^2(3+\beta)=1\\ 3-\beta=9(1+\beta^2)\end{cases} \quad (112)$$

$$\begin{cases}3+\beta=1\\ \alpha^2(3-\beta)=9(1+\beta^2)\end{cases} \quad (113)$$

$$\begin{cases} \alpha^2(3+\beta) = 9(1+\beta^2) \\ 3-\beta = 1 \end{cases} \quad (114)$$

$$\begin{cases} 3+\beta = 9(1+\beta^2) \\ \alpha^2(3-\beta) = 1 \end{cases} \quad (115)$$

$$\begin{cases} \alpha(3+\beta) = 1 \\ \alpha(3-\beta) = 9(1+\beta^2) \end{cases} \quad (116)$$

$$\begin{cases} \alpha(3+\beta) = 9(1+\beta^2) \\ \alpha(3-\beta) = 1 \end{cases} \quad (117)$$

$$\begin{cases} \alpha = 1 \\ (3-\beta)(3+\beta) = 9(1+\beta^2) \end{cases} \quad (118)$$

$$\begin{cases} \alpha^2 = 1+\beta^2 \\ (3-\beta)(3+\beta) = 9 \end{cases} \quad (119)$$

$$\begin{cases} \alpha = 1+\beta^2 \\ \alpha(3-\beta)(3+\beta) = 9 \end{cases} \quad (120)$$

$$\begin{cases} \alpha = 3(1+\beta^2) \\ \alpha(3-\beta)(3+\beta) = 3 \end{cases} \quad (121)$$

$$\begin{cases} \alpha = 9(1+\beta^2) \\ \alpha(3-\beta)(3+\beta) = 1 \end{cases} \quad (122)$$

1. For the system (96), the equation gives a solution, which is contrary to condition (94)$\beta^2 - \beta - 2 = 0\beta = 2, \alpha = 3$
2. For the system (97) gives a decision $\beta = 6\alpha^2 < 0$
3. For the system (98) the solution This solution satisfies condition (94). In this case, γ = 8/5. Hence, k = 8, l = 5. Next, from (33) and (34) we obtain:$\alpha = \frac{3}{2}, \beta = 1$

$$\begin{cases} h = 1, x = 11, a = 44, b = 117, c = 240, d = 125 \\ e = 267, f = 244 \end{cases} \quad (123)$$

So, we found a solution to the system (33). This decision was already known to Euler, who considered him a minimum (and, apparently, he was not wrong), so we'll call it the Euler solution.

4. For the system (99) the solution β = -6. It's impossible.

5. For the system (100) an equation. The discriminant of this equation. So, the equation has a real root, which is not difficult to calculate:$\beta^3 - 3\beta^2 + 10\beta + 24 = 0 Q = \frac{7255}{27} > 0$

$\beta \approx 1 + \sqrt[3]{0{,}392} - \sqrt[3]{32{,}392}$

so

$\alpha \approx \frac{9}{2 - \sqrt[3]{0{,}392} + \sqrt[3]{32{,}392}}$

This contradicts (94).$\alpha < 3$

6. For the system (101) we obtain the equation

$\beta^3 + 3\beta^2 + 10\beta - 24 = 0$

The discriminant is Q = 7255/27> 0, the real root. so$\beta \approx 8 - \sqrt[3]{0{,}392} + \sqrt[3]{32{,}392}$

$\alpha \approx \frac{9}{11 - \sqrt[3]{0{,}392} + \sqrt[3]{32{,}392}}$And this contradicts (94).$\alpha > 1$

7. For the system (102), we obtain α = 3, which contradicts (94): α <3.
8. For the system (103) solution β = 0, α = 1, which is impossible.
9. For the system (104) α = 3, which contradicts (94); α <3.
10. For the system (105) an equation

$\beta^4 - 8\beta^2 - 8 = 0$

Hence, contrary to (94); α <3.$\beta^2 = 4 + \frac{1}{2}\sqrt{96} \alpha = 15 + \frac{3}{2}\sqrt{96}$

11. For the system (106) β = 0, α = 1.Protivorechit (94); α> 1
12. For the system (107) β = 0. It is impossible, because the patched. α must be a rational number.$\alpha = \sqrt{3}$
13. For the system (108) β = 0, α = 1. It is impossible, because α> 1.
14. For the system (109) β = 0, α = 1. It is impossible for the same reason as in the previous case.
15. For the system (110) an equation

$\beta(\beta^2 - 3\beta + 2) = 0$

Here. Means. Solutions and impossible, because contradict (94). Solution gives Euler solution (the same as in the case (98)).$\beta_1 = 0, \beta_2 = 1, \beta_3 = 2\alpha_1 = 1, \alpha_2 = \frac{3}{2}, \alpha_3 = 3\alpha_1 = 1, \beta_1 = 0\alpha_3 = 3, \beta_3 = 2\alpha_2 = \frac{3}{2}, \beta_2 = 1$

16.For the system (111) an equation

$\beta(\beta^2 + 3\beta + 2) = 0$

One non-negative solution: β = 0. Hence, α = 1, which is impossible.

17.For the system (112) the equation

$9\beta^2 + \beta + 6 = 0$ Positives no roots.

18.For the system (113) β = -2. It's impossible.

19.β = 2, α = 3 for a system (114). Contradicts (94).

20.For the system (115) an equation

$9\beta^2 - \beta + 6 = 0$ It has no real roots, because

.$\beta_{1,2} = \frac{1}{18}(1 \pm \sqrt{-215})$

21.For the system (116) we obtain the equation

$9\beta^3 + 27\beta^2 + 10\beta + 24 = 0$ This equation has no positive real roots.

22.For the system (117) we obtain the equation

$9\beta^3 - 27\beta^2 + 10\beta - 24 = 0$

This equation has one real root

$\beta \approx 1 + \sqrt[3]{1{,}75} - \sqrt[3]{5{,}29}$

here

$\alpha \approx \frac{1}{2 - \sqrt[3]{1{,}75} + \sqrt[3]{5{,}29}} < 1$This contradicts (94); α> 1.

23.For the system (118) α = 1, which is impossible.

24.For the system (119) an equation

$\beta^2(\beta^2 - 8) = 0$Which has no rational solutions except β = 0, but in this case α = 1, which is impossible.

25.For the system (120) an equation. A case similar to the previous.$\beta^2(8-\beta^2)=0$

26.For the system (121) an equation that has no rational decisions, because .$\beta^4-8\beta^2-8=0\beta^2=4+\frac{1}{2}\sqrt{96}$

27.For the system (122) an equation that does not rational decisions, because .$9\beta^4-72\beta^2-80=0\beta^2=4+\frac{1}{18}\sqrt{8064}$

Option 9.

$$\begin{cases} x-h=3\alpha l \\ x+h=\frac{k}{\alpha} \\ x-2h=3\beta(k-l) \\ x+2h=\frac{k+l}{\beta} \end{cases} \quad (124)$$

compute

$$\gamma=\frac{\alpha(2+3\alpha\beta)}{3\beta-2\alpha}=\frac{\alpha(9\alpha+6\beta)}{1+6\alpha\beta}$$

The condition γ> 1 gives

$$\begin{cases} \frac{1}{3}<\alpha<1 \\ \beta<\frac{4\alpha}{3(1-\alpha^2)} \end{cases} \quad (125)$$

The equation

$$\frac{2+3\alpha\beta}{3\beta-2\alpha}=\frac{9\alpha+6\beta}{1+6\alpha\beta}$$

equivalent to the equation

$$-9\alpha^2+9\beta^2-9\alpha^2\beta^2=1.$$

Or

$$(3\beta-1)(3\beta+1)=9\alpha^2(1+\beta^2) \quad (126)$$

Equation (126) is split into 30 systems:

$$\begin{cases} (3\beta-1)(3\beta+1)=9 \\ \alpha^2(1+\beta^2)=1 \end{cases} \quad (127)$$

$$\begin{cases} (3\beta-1)(3\beta+1)=9\alpha \\ \alpha(1+\beta^2)=1 \end{cases} \quad (128)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=9\alpha^2\\ 1+\beta^2=1\end{cases} \quad (129)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=3\\ 2\alpha^2(1+\beta^2)=1\end{cases} \quad (130)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=3\alpha\\ 3\alpha(1+\beta^2)=1\end{cases} \quad (131)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=3\alpha^2\\ 3(1+\beta^2)=1\end{cases} \quad (132)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=9(1+\beta^2)\\ \alpha^2=1\end{cases} \quad (133)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=9\alpha(1+\beta^2)\\ \alpha=1\end{cases} \quad (134)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=3(1+\beta^2)\\ 3\alpha^2=1\end{cases} \quad (135)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=3\alpha(1+\beta^2)\\ 3\alpha=1\end{cases} \quad (136)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=1+\beta^2\\ 9\alpha^2=1\end{cases} \quad (137)$$

$$\begin{cases}(3\beta-1)(3\beta+1)=\alpha(1+\beta^2)\\ 9\alpha=1\end{cases} \quad (138)$$

$$\begin{cases}3\beta-1=1\\ 3\beta+1=9\alpha^2(1+\beta^2)\end{cases} \quad (139)$$

$$\begin{cases}3\beta-1=\alpha\\ 3\beta+1=9\alpha(1+\beta^2)\end{cases} \quad (140)$$

$$\begin{cases}3\beta-1=\alpha^2\\ 3\beta+1=9(1+\beta^2)\end{cases} \quad (141)$$

$$\begin{cases}3\beta-1=1+\beta^2\\ 3\beta+1=9\alpha^2\end{cases} \quad (142)$$

$$\begin{cases}3\beta-1=3\\ 9\beta+1=3\alpha^2(1+\beta^2)\end{cases} \quad (143)$$

$$\begin{cases}3\beta-1=3\alpha\\ 3\beta+1=3\alpha(1+\beta^2)\end{cases} \quad (144)$$

$$\begin{cases}3\beta-1=3\alpha^2\\ 3\beta+1=3(1+\beta^2)\end{cases} \quad (145)$$

$$\begin{cases} 3\beta - 1 = 3(1+\beta^2) \\ 3\beta + 1 = 3\alpha^2 \end{cases} \quad (146)$$

$$\begin{cases} 3\beta - 1 = 9 \\ 3\beta + 1 = \alpha^2(1+\beta^2) \end{cases} \quad (147)$$

$$\begin{cases} 3\beta - 1 = 9\alpha \\ 3\beta + 1 = \alpha(1+\beta^2) \end{cases} \quad (148)$$

$$\begin{cases} 3\beta - 1 = 9\alpha^2 \\ 3\beta + 1 = 1+\beta^2 \end{cases} \quad (149)$$

$$\begin{cases} 3\beta - 1 = 9(1+\beta^2) \\ 3\beta + 1 = \alpha^2 \end{cases} \quad (150)$$

$$\begin{cases} 3\beta + 1 = 1 \\ 3\beta - 1 = 9\alpha^2(1+\beta^2) \end{cases} \quad (151)$$

$$\begin{cases} 3\beta + 1 = \alpha \\ 3\beta - 1 = 9\alpha(1+\beta^2) \end{cases} \quad (152)$$

$$\begin{cases} 3\beta + 1 = 3 \\ 3\beta - 1 = 3\alpha^2(1+\beta^2) \end{cases} \quad (153)$$

$$\begin{cases} 3\beta + 1 = 3\alpha \\ 3\beta - 1 = 3\alpha(1+\beta^2) \end{cases} \quad (154)$$

$$\begin{cases} 3\beta + 1 = 9 \\ 3\beta - 1 = \alpha^2(1+\beta^2) \end{cases} \quad (155)$$

$$\begin{cases} 3\beta + 1 = 9\alpha \\ 3\beta - 1 = \alpha(1+\beta^2) \end{cases} \quad (156)$$

Consider these options:

1. For the system (127) an equation that has no rational solution.$9\beta^2 = 10$
2. For the system (128), we obtain an equation which also has no rational solution.$9\beta^4 + 8\beta^2 - 10 = 0$
3. For the system (129) solution β = 0, α = i / 3, which is impossible.
4. For the system (130) solution β = 2/3, that is impossible.$\alpha = \sqrt{3/13}$
5. For the system (131) yields an equation that has no rational decisions.$9\beta^4 + 8\beta^2 - 2 = 0$
6. For the system (132) β = 2i / 3, which is impossible.
7. For the system (133) α = 1, which contradicts (125).

8. For the system (134) is the same as in the previous case.
9. For the system (135) - not a rational decision.$\alpha = \sqrt{1/3}$
10. For the system (136) α = 1/3, which is contrary to condition (125).
11. For the system (137), too, as in the previous case.
12. For the system (138) α = 1/9, contrary to condition (125).
13. For the system (139) solution β = 2/3 - not rational.$\alpha = \sqrt{3/13}$
14. For the system (140) an equation. This equation has no rational roots, because It is not a perfect square.$27\beta^3 - 9\beta^2 + 24\beta - 10 = 0 Q = \frac{12951}{531441}$
15. For the system (141) an equation that has no real solutions:.$9\beta^2 - 3\beta + 8 = 0 \beta_{1,2} = 1/18(3 \pm \sqrt{-279})$
16. For the system (142) we obtain the solution:. But the first decision will lead to the Euler case, since in this case, γ = 8/5 and k = 8, l = 5. The second solution is not rational.$\alpha_1 = \frac{2}{3}, \beta_1 = 1$ и $\alpha_2 = \frac{\sqrt{7}}{3}, \beta_2 = 2$
17. For the system (143) solution, β = 4/3 - not rational.$\alpha = \sqrt{3/5}$
18. For the system (144) we get the equation. Its root. So, contrary to the condition (125).$3\beta^3 - 3\beta^2 - 2 = 0 \beta \approx 0{,}233 + \sqrt[3]{0{,}669} \alpha = \frac{3\beta - 1}{9} \approx \frac{0{,}7 + 3\sqrt[3]{0{,}669}}{9} < \frac{1}{3}$
19. For the system (145), we obtain an equation that has a complex root, which is impossible.$3\beta^2 - 3\beta + 2 = 0 \beta = \frac{1}{6}(3 \pm \sqrt{-15})$
20. For the system (146), we obtain an equation which also has only complex roots.$3\beta^2 - 3\beta + 4 = 0 \beta = \frac{1}{6}(3 \pm \sqrt{-39})$
21. For the system (147) solution, β = 10/3 - not rational.$\alpha = \sqrt{\frac{33}{101}}$
22. For the system (148) we get the equation. The equation has no rational roots because discriminant it is not a perfect square.$3\beta^3 - \beta^2 - 24\beta - 10 = 0 Q = \frac{2324511}{531441}$
23. For the system (149) solution β = 0, α = i / 3 - is impossible.

24.For the system (150) an equation whose solution is complex.$9\beta^2 - 3\beta + 10 = 0\beta = \frac{1}{18}(3 \pm \sqrt{-351})$

25.For the system (151) β = 2/3, α = - not rational solution.$\sqrt{1/13}$

26.For the system (152) yields an equation that has no positive real solutions.$27\beta^3 + 9\beta^2 + 24\beta + 10 = 0$

27.For the system (153) β = 2/3 - not rational.$\alpha = \sqrt{3/13}$

28.For the system (154), we obtain an equation that has no positive real solutions.$3\beta^3 + \beta^2 + 2 = 0$

29.For the system (155) β = 8/3, - a decision not rational.$\alpha = \sqrt{63/73}$

30.For the system (156) we get the equation. The discriminant of this equation, ie equation has three real roots. It is easy to calculate them:.$3\beta^3 + \beta^2 - 24\beta + 10 = 0Q = -\frac{7846713}{64*217^3} < 0\beta_1 = -0{,}506, \beta_2 = 0{,}433, \beta_3 = 0{,}607$

Next, from the (156) follows, and (125) 1/3 <α <1. $\alpha = \frac{3\beta+1}{9} = \frac{3\beta-1}{1+\beta^2}$

Combining these two conditions obtain

$$\begin{cases} 0{,}66 < \beta < 2{,}66 \\ 0{,}47 < \beta < 8{,}53 \\ -\propto < \beta < 1 \\ 2 < \beta < \propto \end{cases}$$

Those. 0,66 <β <1 2 <β <2.66. Obviously, none of the roots does not satisfy these conditions.

Option 10.

$$\begin{cases} x - h = 3\alpha l \\ x + h = \frac{k}{\alpha} \\ x - 2h = \beta(\beta k - l) \\ x + 2h = \frac{3(k+l)}{\beta} \end{cases} \quad (157)$$

compute $\gamma = \frac{\alpha(3+\alpha\beta)}{\beta-3\alpha} = \frac{\alpha(9\alpha+2\beta)}{1+2\alpha\beta}$

The condition γ> 1 gives

$$\begin{cases} \frac{1}{3} < \alpha < 1 \\ \beta < \frac{6\alpha}{1-\alpha^2} \end{cases} \quad (158)$$

Condition

$$\frac{3+\alpha\beta}{\beta-3\alpha} = \frac{9\alpha+2\beta}{1+2\alpha\beta}$$

It gives the equation

$27\alpha^2 - 2\beta^2 + 2\alpha^2\beta^2 + 4\alpha\beta + 3 = 0$, Ie

(159) $3(9\alpha^2 + 1) = 2\beta(\beta - \alpha^2\beta - 2\alpha)$

Equation (159) gives the following possibilities:

$$\begin{cases} 3 = 2\beta \\ 9\alpha^2 + 1 = \beta - \alpha^2\beta - 2\alpha \end{cases} \quad (160)$$

$$\begin{cases} 3 = \beta \\ 9\alpha^2 + 1 = 2(\beta - \alpha^2\beta - 2\alpha) \end{cases} \quad (161)$$

$$\begin{cases} 3 = \beta(\beta^2 - \alpha^2\beta - 2\alpha) \\ 9\alpha^2 + 1 = 2 \end{cases} \quad (162)$$

$$\begin{cases} 3 = \beta - \alpha^2\beta - 2\alpha \\ 9\alpha^2 + 1 = 2\beta \end{cases} \quad (163)$$

$$\begin{cases} 1 = 2\beta \\ 3(9\alpha^2 + 1) = \beta - \alpha^2\beta - 2\alpha \end{cases} \quad (164)$$

$$\begin{cases} 1 = \beta(\beta - \alpha^2\beta - 2\alpha) \\ 3(9\alpha^2 + 1) = 2 \end{cases} \quad (165)$$

$$\begin{cases} 1 = \beta - \alpha^2\beta - 2\alpha \\ 3(9\alpha^2 + 1) = 2\beta \end{cases} \quad (166)$$

$$\begin{cases} 1 = 9\alpha^2 + 1 \\ 3 = 2\beta(\beta - \alpha^2\beta - 2\alpha) \end{cases} \quad (167)$$

$$\begin{cases} 1 = 2\beta(\beta - \alpha^2\beta - 2\alpha) \\ 3(9\alpha^2 + 1) = 1 \end{cases} \quad (168)$$

$$\begin{cases} 3(9\alpha^2 + 1) = 2 \\ 1 = \beta(\beta - \alpha^2\beta - 2\alpha) \end{cases} \quad (169)$$

$$\begin{cases} 3(9\alpha^2 + 1) = \beta - \alpha^2\beta + 2\alpha \\ 1 = 2\beta \end{cases} \quad (170)$$

$$\begin{cases} \beta = 1 \\ 3(9\alpha^2 + 1) = 2(\beta - \alpha^2\beta - 2\alpha) \end{cases} \quad (171)$$

The condition γ> 1 gives

$$\begin{cases} \frac{1}{3} < \alpha < 1 \\ \beta < \frac{6\alpha}{1-\alpha^2} \end{cases} \quad (173)$$

A total of 12 options appears:

1. For the system (160) solution β = 3/2, α = 1/7, contrary to condition (173).
2. For the system (161) solution β = 3 is not rational. $\alpha = \frac{1}{30}(-4 \pm \sqrt{316})$
3. For the system (162) α13 decision contradicts the condition (173).
4. For the system (164) solution β = 1/2 gives an equation that has no positive real solutions. $55\alpha^2 + 4\alpha + 5 = 0$
5. For the system (165) decision α = i / not valid. $\sqrt{27}$
6. For the system (163) an equation, which obviously has no positive real roots. $3\alpha^4 - 2\alpha^2 + 4\alpha + 5 = 0$
7. For the system (166) we get the equation. The roots of this equation does not satisfy the condition (173). $27\alpha^4 - 24\alpha^2 + 4\alpha - 1 = 0$ $\alpha_1 = -1{,}2374, \alpha_2 = -0{,}8547, \alpha_3 = 1{,}0113, \alpha_4 = 1{,}0809$
8. For the system (167) α = 0 the solution - it is impossible.
9. For the system (168) the decision - not valid. $\alpha = 1/3\sqrt{2/3}i$
10. For the system (169) the decision - not valid. $\alpha = \frac{1}{3\sqrt{3}}i$
11. For the system (170) solution β = 1/2, and α - the root of the equation that has no positive real roots. $55\alpha^2 + 4\alpha + 1 = 0$
12. For the system (171) solution β = 1 and α - root of the equation, which also has no positive roots. $29\alpha^2 + 4\alpha + 1 = 0$

Option 11.

$$\begin{cases} x - h = 3\alpha l \\ x + h = \frac{k}{l} \\ x - 2h = 3\beta(k + l) \\ x + 2h = \frac{k-l}{\beta} \end{cases} \quad (174)$$

compute

$\gamma = \frac{\alpha(3\alpha\beta-2)}{3\beta-2\alpha} = \frac{\alpha(9\alpha-6\beta)}{1+6\alpha\beta}$

The condition γ> 1 gives

$\begin{cases} \alpha > 1 \\ \beta < \frac{9\alpha^2-1}{12\alpha} \end{cases}$ (175)

The equation

$\frac{3\alpha\beta-2}{3\beta-2\alpha} = \frac{9\alpha-6\beta}{1+6\alpha\beta}$

equivalent to the equation

$9\alpha^2 + 9\beta^2 + 9\alpha^2\beta^2 - 24\alpha\beta = 1$, or

$(3\alpha - 3\beta)^2 + (3\alpha\beta - 1)^2 = 2$

Hence there are three options:

$\begin{cases} |3\alpha - 3\beta| = 2 \\ |3\alpha\beta - 1| = 0 \end{cases}$ (176)

$\begin{cases} |3\alpha - 3\beta| = 1 \\ |3\alpha\beta - 1| = 1 \end{cases}$ (177)

$\begin{cases} |3\alpha - 3\beta| = 0 \\ |3\alpha\beta - 1| = 2 \end{cases}$ (178)

It is easy to calculate that in all three embodiments, α = 1, which contradicts (175).

Option 12.

$\begin{cases} x - h = 3\alpha l \\ x + h = \frac{k}{\alpha} \\ x - 2h = \beta(k - l) \\ x + 2h = \frac{3(k-l)}{\beta} \end{cases}$ (179)

compute

$\gamma = \frac{\alpha(\alpha\beta-2)}{\beta-2\alpha} = \frac{\alpha(9\alpha-2\beta)}{2\alpha\beta+1}$

The condition γ> 1 gives

$\begin{cases} \alpha > 1 \\ \beta < \frac{9\alpha^2-2\beta}{4\alpha} \end{cases}$ (180)

The equation

$$\frac{\alpha\beta-2}{\beta-2\alpha}=\frac{9\alpha-2\beta}{2\alpha\beta+1}$$

equivalent to the equation

$9\alpha^2+\beta^2+\alpha^2\beta^2-8\alpha\beta=1$, or

$(3\alpha-\beta)^2+(\alpha\beta-1)^2=2.$

As in the embodiment 11, there are three possibilities:

$$\begin{cases}|3\alpha-\beta|=0\\|\alpha\beta-1|=2\end{cases}\ (181)$$

$$\begin{cases}|3\alpha-\beta|=1\\|\alpha\beta-1|=1\end{cases}\ (182)$$

$$\begin{cases}|3\alpha-\beta|=2\\|\alpha\beta-1|=0\end{cases}\ (183)$$

As in the previous case, the decision for each case of α = 1, which contradicts (180).

Option 13.

$$\begin{cases}x-h=\alpha l\\x+h=\frac{3k}{l}\\x-2h=3\beta(k-l)\\x+2h=\frac{k+l}{\beta}\end{cases}\ (184)$$

compute

$$\gamma=\frac{\alpha(2+\alpha\beta)}{9\beta-2\alpha}=\frac{\alpha(\alpha+2\beta)}{3+2\alpha\beta}$$

The condition $\gamma > 1$ gives

$$\begin{cases}\sqrt{3}<\alpha<3\\\beta<\frac{4\alpha}{9-\alpha^2}\end{cases}\ (185)$$

The equation

$$\frac{2+\alpha\beta}{9\beta-2\alpha}=\frac{\alpha+2\beta}{3+2\alpha\beta}$$

equivalent to the equation

$\alpha^2\beta^2+\alpha^2-9\beta^2+\alpha\beta+3=0$, or

$\alpha^2\beta^2+\alpha\beta+3=(3\beta-\alpha)(3\beta+\alpha).$

Hence we obtain the system

$$\begin{cases} 3\beta - \alpha = 1 \\ 3\beta + \alpha = \alpha^2\beta^2 + \alpha\beta + 3 \end{cases}, \text{ or}$$

$\alpha^4 + 2\alpha^3 - 5\alpha^2 - 15\alpha + 27 = 0$ (186)

Option 14.

$$\begin{cases} x - h = \alpha l \\ x + h = \frac{3k}{\alpha} \\ x - 2h = \beta(k - l) \\ x + 2h = \frac{3(k+l)}{\beta} \end{cases} \quad (187)$$

compute

$$\gamma = \frac{\alpha(6+\alpha\beta)}{9\beta-6\alpha} = \frac{\alpha(3\alpha+2\beta)}{3+2\alpha\beta}.$$

The condition $\gamma > 1$ gives

$$\begin{cases} 1 < \alpha < 3 \\ \beta < \frac{12\alpha}{9-\alpha^2} \end{cases} \quad (188)$$

The equation

$$\frac{6+\alpha\beta}{9\beta-6\alpha} = \frac{3\alpha+2\beta}{3+2\alpha\beta}$$

equivalent to the equation

$-9\alpha^2 + 9\beta^2 - \alpha^2\beta^2 = 9$, or

$9(\beta - 1)(\beta + 1) = \alpha^2(9 + \beta^2)$ (189)

Equation (189) is split into 19 systems:

$$\begin{cases} 9(\beta - 1) = 1 \\ \beta + 1 = \alpha^2(9 + \beta^2) \end{cases} \quad (190)$$

$$\begin{cases} 9(\beta + 1) = 1 \\ \beta - 1 = \alpha^2(9 + \beta^2) \end{cases} \quad (191)$$

$$\begin{cases} 9(\beta - 1) = \alpha \\ \beta + 1 = \alpha(9 + \beta^2) \end{cases} \quad (192)$$

$$\begin{cases} 9(\beta - 1) = \alpha^2 \\ \beta + 1 = 9 + \beta^2 \end{cases} \quad (193)$$

$$\begin{cases} 9(\beta + 1) = \alpha \\ \beta - 1 = \alpha(9 + \beta^2) \end{cases} \quad (194)$$

$$\begin{cases} 9(\beta - 1) = 9 + \beta^2 \\ \beta + 1 = \alpha^2 \end{cases} (195)$$

$$\begin{cases} 9(\beta + 1) = \alpha \\ \beta - 1 = \alpha(9 + \beta^2) \end{cases} (196)$$

$$\begin{cases} 9(\beta + 1) = \alpha^2 \\ \beta - 1 = 9 + \beta^2 \end{cases} (197)$$

$$\begin{cases} 9(\beta + 1) = 9 + \beta^2 \\ \beta - 1 = \alpha^2 \end{cases} (198)$$

$$\begin{cases} 3(\beta - 1) = 1 \\ 3(\beta + 1) = \alpha^2(9 + \beta^2) \end{cases} (199)$$

$$\begin{cases} 3(\beta - 1) = \alpha \\ 3(\beta + 1) = \alpha(9 + \beta^2) \end{cases} (200)$$

$$\begin{cases} 3(\beta - 1) = \alpha^2 \\ 3(\beta + 1) = 9 + \beta^2 \end{cases} (201)$$

$$\begin{cases} 3(\beta + 1) = 1 \\ 3(\beta - 1) = \alpha^2(9 + \beta^2) \end{cases} (202)$$

$$\begin{cases} 3(\beta + 1) = \alpha \\ 3(\beta - 1) = \alpha(9 + \beta^2) \end{cases} (203)$$

$$\begin{cases} 3(\beta + 1) = \alpha^2 \\ 3(\beta - 1) = 9 + \beta^2 \end{cases} (204)$$

$$\begin{cases} (\beta - 1)(\beta + 1) = 1 \\ 9 = \alpha^2(9 + \beta^2) \end{cases} (205)$$

$$\begin{cases} (\beta - 1)(\beta + 1) = \alpha \\ 9 = \alpha(9 + \beta^2) \end{cases} (206)$$

$$\begin{cases} (\beta - 1)(\beta + 1) = \alpha^2 \\ 9 = 9 + \beta^2 \end{cases} (207)$$

$$\begin{cases} (\beta - 1)(\beta + 1) = 9 + \beta^2 \\ 9 = \alpha^2 \end{cases} (208)$$

1. For the system (190) solution, β = 10/9 - not rational.$\alpha = \frac{\sqrt{19}}{9}$
2. For the system (191) solution β = -8 / 9 - negative.
3. For the system (192) an equation.$9\beta^3 - 9\beta^2 + 80\beta - 82 = 0$

4. For the system (193) an equation whose solution is imaginary.$\beta^2 - \beta + 8 = 0$
5. For the system (194), we obtain an equation that has no positive solutions.$\beta^3 + \beta^2 + 18 = 0$
6. For the system (195) solution, ie, And we get the Euler solution.$\alpha_1 = \sqrt{7}, \beta_1 = 6, \alpha_2 = 2, \beta_2 = 3\gamma = \frac{2(6+6)}{27-12} = \frac{8}{5}$
7. For the system (196), we obtain an equation that has no positive solutions.$9\beta^3 + 9\beta^2 + 80\beta + 82 = 0$
8. For the system (197) an equation that has no real solutions.$\beta^2 - \beta + 8 = 0$
9. For the system (198) the decision - not rational.$\alpha_1 = i, \beta_1 = 0, \alpha_2 = 2\sqrt{2}, \beta_2 = 9$
10. For the system (199) the decision - not rational.$\alpha = \sqrt{63/97}, \beta = 4/3$
11. For the system (200) we get the equation.$\beta^3 - \beta^2 + 8\beta - 10 = 0$
12. For the system (201), we obtain equation whose solution imaginary.$\beta^2 - 3\beta + 6 = 0$
13. For the system (202) solution, β = -2 / 3 - imaginary.$\alpha = \frac{9i}{\sqrt{85}}$
14. For the system (203), we obtain an equation for which there are no positive decisions.$\beta^3 + \beta^2 + 8\beta + 10 = 0$
15. For the system (204) the equation. Solution imaginary.$\beta^2 - 3\beta + 12 = 0$
16. For the system (205) the decision - not rational.$\beta = \sqrt{2}$
17. For the system (206) an equation. The decision - not rational.$\beta^4 + 8\beta^2 - 18 = 0 \beta^2 = 1/2(-8 + \sqrt{136})$
18. For the system (207) solution β = 0, α = i - imaginary.
19. The system (208) has no solution.

Option 15.

$$\begin{cases} x-h=\alpha l \\ x+h=\frac{3k}{\alpha} \\ x-2h=3\beta(k+l) \\ x+2h=\frac{k-l}{\beta} \end{cases} \quad (209)$$

compute

$\gamma=\frac{\alpha(2-\alpha\beta)}{2\alpha-9\beta}=\frac{\alpha(\alpha-2\beta)}{1+2\alpha\beta}$.

The condition γ> 1 gives

$$\begin{cases} 1<\alpha<3 \\ \beta<\frac{\alpha^2-1}{4\alpha} \end{cases} \quad (210)$$

The equation

$\frac{2-\alpha\beta}{2\alpha-9\beta}=\frac{\alpha-2\beta}{1+2\alpha\beta}$

equivalent to the equation

$\alpha^2+9\beta^2-8\alpha\beta+\alpha^2\beta^2=1$, or

$(\alpha-3\beta)^2+(\alpha\beta-1)^2=2$ (211)

The equation (211) yields the system of equations

$$\begin{cases} |\alpha-3\beta|=1 \\ |\alpha\beta-1|=1 \end{cases}$$

Which has seven decisions:

$\alpha_1=0, \beta_1=\frac{1}{3}$; $\alpha_2=-1, \beta_2=0$; $\alpha_3=3, \beta_3=\frac{2}{3}$; $\alpha_4=8, \beta_4=3$; $\alpha_5=-7, \beta_5=-2$; $\alpha_6=0, \beta_6=-\frac{1}{3}$; $\alpha_7=1, \beta_7=0$. None of them does not satisfy condition (210).

Option 16.

$$\begin{cases} x-h=\alpha l \\ x+h=\frac{3k}{\alpha} \\ x-2h=\beta(k+l) \\ x+2h=\frac{3(k-l)}{\beta} \end{cases} \quad (212)$$

compute

$\gamma=\frac{\alpha(6-\alpha\beta)}{6\alpha-9\beta}=\frac{\alpha(3\alpha-2\beta)}{3+2\alpha\beta}$.

From the condition γ> 1 we obtain

$$\begin{cases} 1 < \alpha < 3 \\ \beta < \frac{3(\alpha^2-1)}{4\alpha} \end{cases} (213)$$

The equation

$$\frac{6-\alpha\beta}{6\alpha-9\beta} = \frac{3\alpha-2\beta}{3+2\alpha\beta}$$

equivalent to the equation

$9\alpha^2 + 9\beta^2 - 24\alpha\beta + \alpha^2\beta^2 = 9$, or

$9(\alpha-\beta)^2 + (\alpha\beta-3)^2 = 18$. (214)

The solution of equation (214) is obtained from equations

$$\begin{cases} |\alpha-\beta| = 1 \\ |\alpha\beta-3| = 3 \end{cases} \quad (215)$$

This system has eight decisions:

$\alpha_1 = 3, \beta_1 = 2;\ \alpha_2 = -2, \beta_2 = -3;\ \alpha_3 = 0, \beta_3 = -1;\ \alpha_4 = 1, \beta_4 = 0;\ \alpha_5 = -3, \beta_5 = -2;\ \alpha_6 = 2, \beta_6 = 3;\ \alpha_7 = 0, \beta_7 = 1;\ \alpha_8 = -1, \beta_8 = 0.$ From these solutions only satisfies condition (213), but in this case, γ = 0.$\alpha_6 = 2, \beta_6 = 3$

Thus, it is proved that the Euler brick does not exist.

The author is grateful to Nikita Klyopov for his invaluable help in writing this article.

Literature

1. Gardner M. Tic-Tac-Toe. - M .: Mir, 1988.

2. W. Ball, Coxeter, Mathematical essays and entertainment. - M .: Mir, 1986.

3. Lander LJ, Parkin TR, Selfridge JL Mathematics of Computation, 1967, vol. XXI, pp. 446 - 459.