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VARIATIONAL ITERATION METHOD FOR ANOMALOUS DIFFUSION PROCESSES

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***Abstract.** Approximate solutions to anomalous diffusion processes are represented by using the theory of fractional variations, Caputo. The variational iterative method has been successfully extended to the case of solving differential equations with fractional derivatives and the Lagrange multiplier of the method is identified more accurately.*

***Keywords:** fractional derivative, Caputo derivative, variational iteration method, anomalous diffusion, Laplace transform.*

Вариационный итерационный метод для аномальных диффузионных процессов

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***Аннотация.** Приближенные решения аномальных диффузионных процессов решено с помощью теории дробных вариаций, Капута. Вариационный итерационный метод успешно распространен на случай решения дифференциальных уравнений с дробными производными, и множитель Лагранжа метода идентифицируется более точно.*

***Ключевые слова:** дробная производная, производная Капута, вариационный итерационный метод, аномальная диффузия, преобразование Лапласа.*

Fractional calculus is a powerful tool for finding solutions to nonlinear problems in various applications where successful fractional models for sorption of metal, fractional operation, sub-diffusion processes, frequency-dependent attenuation, etc. have been developed. Simultaneously, approximate solutions of fractional differential equations (FDEs) are intensively developing. It was proved that the variational iterative method (VIM) is an effective nonlinear method for

solving ordinary differential equations (ODE). Some efforts have been made to solve the FDEs, but not very successfully, due to the difficulty of determining the Lagrange multiplier. In fact, the Lagrange multiplier of the method can be explicitly identified using the Laplace transform. For clarity of the following expression, we briefly dwell on the VIM methodology for solving the following FDE:

$${}_0^C D_t^\alpha u(t) + R[u(t)] + N[u(t)] = k(t), \quad \alpha > 0, \quad (1)$$

where ${}_0^C D_t^\alpha$ is the left Caputo derivative, R is the linear operator, L is the nonlinear operator, and $k(t)$ is the given continuous function. Then we propose some new variational iterative formulas and study some fractional diffusion equations describing the flow in a porous medium.

Definition 1 Derived Caputo is defined as

$${}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\xi)^{\alpha-m+1}} \frac{d^m}{d\xi^m} u(\xi) d\xi. \quad (2)$$

where $u(t) \in C^m[0, T]$ and Γ is the gamma function, with $C^m[0, T]$ space of functions, which are m times continuously differentiable on the interval $[0, T]$.

Definition 2 The α -th Riemann-Liouville derivative of a function $u(t)$ is defined as

$${}_0^{RL} D_t^\alpha u = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt} \right)^m \int_0^t \frac{1}{(t-\xi)^{\alpha-m+1}} u(\xi) d\xi, \quad \alpha > 0, \quad m = [\alpha] + 1, \quad t > 0, \quad (3)$$

Definition 3 Riemann-Liouville integration of α order is defined as

$${}_0 I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} u(\xi) d\xi, \quad \alpha > 0. \quad (4)$$

The Laplace transform of the original function $u(t)$ of a real variable t , is determined by the integral as

$$\bar{u}(s) = L[u(t)] = \int_0^{\infty} e^{-st} u(t) dt, \quad t > 0,$$

where the parameter s is a complex number.

Proposition 1. There is a Laplace transform of the term ${}^C D_t^\alpha u$

$$L[{}^C D_t^\alpha u] = s^\alpha \bar{u}(s) - \sum_{k=0}^{m-1} u^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \leq m. \quad (5)$$

Proposition 2 Transformation of the Laplace derivative

$$L[{}^{RL} D_t^\alpha u] = s^\alpha \bar{u}(s) - \sum_{k=0}^{m-1} u^{(\alpha-k-1)}(0^+) s^k, \quad m-1 < \alpha \leq m.$$

The inverse Laplace transform is determined by the complex integral as

$$u(t) = L^{-1}[u(s)] = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} \bar{u}(s) ds,$$

where integration is performed along the ordinate axis $\operatorname{Re}(s) = \gamma$ in the complex plane, so that γ more than the real part of all the features $\bar{u}(s)$.

The variational iteration method has ever been extended to FDE. Since such integration by parts from the calculus cannot be performed, the extension mainly used the Lagrange multipliers in ordinary differential equations. For example, the so-called Lagrange multiplier $\lambda(t, \xi) = -1$ was used and the formula for the variational iteration of equation (1) was given as

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t, \xi) ({}^C D_\xi^\alpha u_n + R[u_n] + N[u_n] - k(\xi)) d\xi, & 0 < \alpha < 1, \\ \lambda(t, \xi) = -1. \end{cases} \quad (6)$$

we can verify that the convergence of the above iterative formula is bad for linear FDE. Then we propose the following iterative formula:

$$u_{n+1} = u_n + \int_0^t \lambda(t, \xi) ({}^C D_\xi^\alpha u_n + R[u_n] + N[u_n] - k(\xi)) d\xi, \quad \alpha > 0, \quad (7)$$

where $\lambda(t, \xi) = (-1)^\alpha (\xi - t)^{\alpha-1} / \Gamma(\alpha)$ is the Lagrange multiplier or a weighted function for any fractional order $\alpha > 0$.

The diffusion process was discussed in many real physical systems, such as a highly branched medium in a porous system, anomalous diffusion in a fractal medium, and heat transfer close to equilibrium. The flow through a porous medium can be better described by a fractional model than by a classical one, since it includes memory effects caused by obstacles in the structure. Readers cite recent publications. In this section, three types of equations of fractional time diffusion are analytically investigated by VIM.

Example Consider a time-fractional diffusion equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial(F(x)u(x,t))}{\partial x}, \quad 0 < \alpha \leq 1. \quad (8)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative, with initial condition $u(x,0) = x^2$.

We replace the fractional Caputo derivative with the local fractional derivative in Eq. (8), and assume $c=1$, $F(x) = -x$ which leads to

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial(xu(x,t))}{\partial x}, \quad 0 < \alpha \leq 1,$$

From iterative formula (7) we have

$$\begin{aligned} u_{n+1} &= u_n - {}_0 I_t^\alpha ({}_0^C D_\xi^\alpha u_n - u_{n,xx}) = \\ u_n &+ \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(t, \xi) \left\{ \frac{\partial^\alpha u_n(x, \xi)}{\partial \xi^\alpha} - \frac{\partial^2 \tilde{u}_n(x, \xi)}{\partial x^2} - \frac{\partial(x\tilde{u}_n(x, \xi))}{\partial x} \right\} (d\xi)^\alpha = \\ u_n &- \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ \frac{\partial^\alpha u_n(x, \xi)}{\partial \xi^\alpha} - \frac{\partial^2 u_n(x, t)}{\partial x^2} - \frac{\partial(xu_n(x, t))}{\partial x} \right\} (d\xi)^\alpha. \end{aligned} \quad (9)$$

Starting from the initial iteration $u_0 = u(x,0) = x^2$, successive approximate solutions can be obtained as

$$u_1 = u_0 - {}_0I_t^\alpha ({}_0^C D_\xi^\alpha u_0 - u_{0,xx}) = x^2 - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left\{ \frac{\partial^\alpha u_0(x, \xi)}{\partial \xi^\alpha} - \frac{\partial^2 u_0(x, \xi)}{\partial x^2} - \frac{\partial(xu_0(x, \xi))}{\partial x} \right\} (d\xi)^\alpha = x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)}, \quad (10a)$$

$$u_2 = u_1 - {}_0I_t^\alpha ({}_0^C D_\xi^\alpha u_1 - u_{1,xx}) = x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad (10b)$$

$$u_3 = x^2 + \frac{(2+3x^2)t^\alpha}{\Gamma(1+\alpha)} + \frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(26+27x^2)t^{3\alpha}}{\Gamma(1+3\alpha)}. \quad (10c)$$

For $n \rightarrow \infty$, $u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{k^i t^{i\alpha}}{\Gamma(1+i\alpha)} = E_\alpha(kt^\alpha)$, is an exact solution to equation (8). Where $k^i = x^2 + (1+x^2)(3^i - 1)$ and $E_\alpha(kt^\alpha)$ is the famous Mittag-Leffler function.

The method of fractional variational iterations in this paper makes it easy to develop very good approximate analytical solutions of fractional models, which turned out to be an effective tool for solving differential equations with fractional derivatives, since the Lagrange multiplier can be more accurately determined using the theory of fractional variations.

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